EXAM ESTIMATION - DETECTION

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Exercise 1: Estimation

The Weibull distribution $\mathcal{W}(k, \theta)$, with shape parameter $k > 0$ and scale parameter $\theta > 0$, is widely used to model the time until failure of life-limited components. Consider n independent identically distributed random variables $X_1, \cdots, X_N$ with law $\mathcal{W}(k, \theta)$ and suppose that k is known.

1. Maximum likelihood estimation
   (a) Express the likelihood of n observation $(x_1, \cdots, x_n)$ and derive the maximum likelihood estimator $\hat{\theta}_{ML}$ for θ.
   (b) Using the change of variable $U = X^k$, show that U is distributed according to a Gamma law with parameters $k = 1$ and θ, $U \sim \mathcal{G}(1, \theta)$, and determine the bias and variance of $\hat{\theta}_{ML}$. Is $\hat{\theta}_{ML}$ unbiased and convergent?
   (c) Determine the Cramer-Rao bound for an unbiased estimator of θ. Is $\hat{\theta}_{ML}$ the efficient estimator for θ?
2. Method of moments
   (a) Derive an estimator for θ using the first moment of $U = X^k$, denoted by $\hat{\theta}_{m_1}$. Is $\hat{\theta}_{m_1}$ biased, convergent, efficient?
   (b) Derive an estimator for θ using the second moment of $U = X^k$, denoted by $\hat{\theta}_{m_2}$.
   (c) Now let $n = 1$ and determine the bias, variance and mean squared error of $\hat{\theta}_{m_2}$.
3. Bayesian estimation with inverse Gamma prior $\mathcal{IG}(\xi, \tau)$ for θ, $\theta \sim \mathcal{IG}(\xi, \tau)$, using the sample $(u_1, \ldots, u_N)$ obtained by the change of variables $U = X^k$:
   (a) Derive the posterior law $p(\theta|u_1, \cdots, u_n; k, \xi, \tau)$, show that it is $\mathcal{IG}(a, b)$ and determine its parameters.
   (b) Derive the MAP estimator $\hat{\theta}_{MAP}$ for θ.
   (c) Show that the expectation of an inverse Gamma random variable $Z \sim \mathcal{IG}(a, b)$ is given by $\mathbb{E}[Z] = \frac{b}{a-1}$ and determine the MMSE estimator $\hat{\theta}_{MMSE}$ for θ.

Exercise 2: Detection

We consider n independent identically distributed random variables $X_1, \cdots, X_n$ from a Lévy law $\mathcal{L}(\mu, \gamma)$ with known location parameter μ.

1. We want to test the hypotheses $H_0 : \gamma = \gamma_0$ and $H_1 : \gamma = \gamma_1$ with $\gamma_1 > \gamma_0$.
   (a) Determine the test statistic of the Neyman-Pearson test, denoted by T.
   (b) Determine the law of T under H_0 and under H_1:
      i. Show that if $X \sim \mathcal{L}(\mu, \gamma)$, then $Y = \frac{1}{X-\mu} \sim \mathcal{G}(\frac{1}{2}, \frac{2}{\gamma})$. *(hint: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$).*
      ii. Using the result of i. and the characteristic function of the Gamma distribution, show that $T \sim \mathcal{G}(\frac{n}{2}, \frac{2}{\gamma})$. Precise the law of T under H_0 and under H_1.
   (c) Determine the integral equation for the significance α of the test. Express the critical value t_α using the cumulative distribution function of the law of T under H_0.
   (d) Determine the expression for the power π of the test.
2. We want to test the hypotheses $H_0 : \gamma = \gamma_0$ and $H_1 : \gamma \neq \gamma_0$. Determine the test statistic of the generalized likelihood ratio test.
3. When k is large, the law $\mathcal{G}(k, \theta)$ can be approximated by a Normal law $\mathcal{N}(\nu, \sigma^2)$. We want to test if we can use this approximation for the test statistic T, i.e., we want to test the hypotheses

$$H_0 : T \sim \mathcal{N}(\nu, \sigma^2), \qquad H_1 : \text{ not } H_0.$$

   Suppose that $n = 32$ and that we have $M = 25$ observations of the test statistic T, denoted t_m, $m = 1, \ldots, M$, given by [1]: $(t_1, \ldots, t_M) = (20,\ 28,\ 16,\ 36,\ 36,\ 68,\ 29,\ 20,\ 28,\ 20,\ 24,\ 24,\ 27,\ 36,\ 52,\ 56,\ 28,\ 20,\ 36,\ 64,\ 20,\ 12,\ 60,\ 12,\ 28)$
   (a) Which test is appropriate for this problem and why?
   (b) Calculate the maximum likelihood estimate for θ for the sample $(t_1, \ldots, t_M)$ and use it to determine ν and σ^2.
   (c) Define the classes for the test with $K = 4$ equi-probable classes (note: $F^{-1}(0.75) = 0.675$, where F is the cumulative distribution function of the standard Normal distribution).
   (d) Perform the test for $\alpha = 0.1$. The quantiles of the chi-square distribution with N degrees of freedom are given by:

N	1	2	3	4	5
$\left(\chi^2_N\right)^{-1}(0.9)$	2.71	4.61	6.25	7.78	9.24

- *Weibull distribution $\mathcal{W}(\alpha, \beta)$:* $\alpha > 0,\ \beta > 0,\ x > 0$
  - *density* $f(x) = \alpha \frac{x^{\alpha-1}}{\beta} \exp\left(-\frac{x^\alpha}{\beta}\right)$
- *Gamma distribution $\mathcal{G}(k, \theta)$:* $k > 0,\ \theta > 0,\ x > 0$
  - *density* $f(x; k, \theta) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} \exp\left(-\frac{x}{\theta}\right)$
  - *mean* $m = k\theta$, *variance* $\nu^2 = k\theta^2$
  - *characteristic function* $\varphi_X(t) = \mathbb{E}\left[e^{itX}\right] = (1 - it\theta)^{-k}$
- *Inverse Gamma distribution $\mathcal{IG}(a, b)$:* $a > 0,\ b > 0,\ x > 0$
  - *density* $f(x; a, b) = \frac{b^a}{\Gamma(a)} x^{-a-1} \exp\left(-\frac{b}{x}\right)$
- *Lévy distribution $\mathcal{L}(\mu, \gamma)$:* $\mu \in \mathbb{R},\ \gamma > 0,\ x \geq \mu$
  - *density* $f(x) = \sqrt{\frac{\gamma}{2\pi}} \frac{\exp\left(-\frac{\gamma}{2(x-\mu)}\right)}{(x-\mu)^{\frac{3}{2}}}$

[1] *For convenience, the values of the continuous random variables X_i have been rounded here.*