A generalized vertex method for computing with fuzzy intervals

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***Abstract*— We introduce a new method for computing functions of fuzzy intervals under various monotonicity assumptions on the concerned functions. Our method makes exact computation for all possibility degrees, without resorting to α-cuts. We formally present the notion of left and right profiles of fuzzy intervals as a tool for fuzzy interval computation. Several results show that interval analysis methods can be directly adapted to fuzzy interval computation where end point of intervals are changed into left and right profiles. Our approach is illustrated by numerous simple examples all along the paper, and a special section is devoted to the application of these concepts to different known problems.**

I. Introduction

In interval computation, the basic problem is: given a function $f(x_1, \cdots, x_n)$ and n intervals $[x_i^-, x_i^+]$, find the interval range of the variable $y = f(x)$ such that $x \in \times_i[x_i^-, x_i^+]$ [1]. Modeling possible values of variables by means of real intervals accounts for some uncertainty, but we can be more precise by modeling uncertainty on a variable x_i by means of a fuzzy interval X_i. Then one way to compute the possible fuzzy range Y of y is to decompose the problem in terms of α-cuts and then to apply a standard interval analysis method. This process has drawbacks: it computes only an approximation of Y, and for each α-cut, the interval algorithm has to be completely executed.

The goal of interval computation is to find the minimum and the maximum of the function when the different possible values of the variables x_i range in their intervals $[x_i^-, x_i^+]$. Some methods are based on finding a finite set of points (called configurations or poles) on which this minimum and maximum is attained [2]. This is the idea of the vertex method [3]. We want to generalize this idea to the fuzzy case without resorting to α-cuts, and we give exact results for all possibility degrees.

In this paper, we propose an approach to the fuzzy problem, based on a particular representation of fuzzy intervals. This representation enables computation on fuzzy intervals under different monotonicity assumptions on the function, using the set of what we call fuzzy configurations.

II. A Refresher On Classical Interval Computation

With n intervals $[x_i^-, x_i^+]$, we call real configuration an element of the set $\mathcal{X} = \times_i[x_i^-, x_i^+]$. Among configurations of $\mathcal{X}$, let us distinguish the *extreme* ones, ie the set $\mathcal{H} = \times_i\{x_i^-, x_i^+\}$. The notion of configuration has been proposed by Buckley for the scheduling problem [4], but in the literature extreme configurations are also called poles [2].

Under some assumption, the maximum of f over $\mathcal{X}$ is actually equal to the maximum of f on $\mathcal{H}$, or on a subset $\mathcal{C} \subseteq \mathcal{H}$. An element $\omega \in \mathcal{H}$ has the form $\omega = (x_1^{\epsilon_1}, \cdots, x_n^{\epsilon_n})$, with $\epsilon_i \in \{+, -\}$. To use this idea, we should write two propositions based on some assumed monotony of the considered function. Before, let us give several definitions of monotony, useful for these propositions. In these definitions, f is a function from $\mathbb{R}^n$ to $\mathbb{R}$.

Definition 1: f is said to be *increasing with respect to* x_i (respectively decreasing) if for all n-tuple $(a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n) \in \mathbb{R}^{n-1}$ the restricted function from $\mathbb{R}$ to $\mathbb{R}$ $f(a_1, a_2, \cdots, a_{i-1}, x_i, a_{i+1}, \cdots, a_n)$ is increasing (respectively decreasing).

Definition 2: f is said *monotonic with respect to each* x_i if for each variable x_i, f is either increasing or decreasing according to x_i.

Definition 3: f is said *locally monotonic with respect to each* x_i if for each variable x_i, for all n-tuple $(a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n) \in \mathbb{R}^{n-1}$ the restricted function $f(a_1, a_2, \cdots, a_{i-1}, x_i, a_{i+1}, \cdots, a_n)$ is monotonic.

In this last definition we should note that f can be increasing for one tuple and decreasing for another. Therefore, a function locally monotonic with respect to each argument is not monotonic in the usual sense (Definition 2). A function is locally monotonic with respect to x_i if the sign of its partial derivative $\frac{\partial f}{\partial x_i}$ does not depend on x_i.

We can now state a well-known proposition:

Proposition 1: Let $x = (x_1, x_2, \cdots, x_n)$ be a tuple of n variables such that $x_i \in [x_i^-, x_i^+]$, and $y = f(x_1, \cdots, x_n) = [y^-, y^+]$. If f is locally monotonic with respect to each argument, then $y^- = min_{\omega\in\mathcal{H}}(f(\omega))$ and $y^+ = max_{\omega\in\mathcal{H}}(f(\omega))$

Proposition 1 enables computations on functions to be performed under a condition of local monotony. This proposition is the basis of the (FWA) Algorithm [3] which computes the fuzzy weighted average. We recall another result which decreases the number of fuzzy configurations used for the computation of a function f with stronger monotony conditions:

Proposition 2: Under the assumption of Proposition 1, if f is locally monotonic with respect to each argument, and $\forall j \in E_1$, f is increasing according to x_j and $\forall j \in E_2$, f is decreasing according to x_j, then

$$y^- = min_{\omega \in \mathcal{H}} \left(f(\omega) | \begin{array}{l} \forall j \in E_1, \omega_j = x_j^- \\ \forall j \in E_2, \omega_j = x_j^+ \end{array} \right)$$

and $y^+ = max_{\omega \in \mathcal{H}} \left(f(\omega) | \begin{array}{l} \forall j \in E_1, \omega_j = x_j^+ \\ \forall j \in E_2, \omega_j = x_j^- \end{array} \right)$

In this paper, we generalize the notion of configuration to fuzzy interval problems, and we give counterparts to the previous propositions.

III. Intuitive Approach

Fuzzy intervals are defined as follow [5]:

Definition 4: A fuzzy interval I, defined by its membership function $\mu_I(.)$ is a fuzzy set such that:
$\forall (x, y, z) \in \mathbb{R}^3 \; z \in [x, y] \Longrightarrow \mu_I(z) \geq min(\mu_I(x), \mu_I(y))$
I is said normalized iff $\exists y \in \mathbb{R}$ such that $\mu_I(y) = 1$
In this paper we only work with normal fuzzy intervals and with upper semi-continuous (USC) membership functions. Note that the α-cut of a fuzzy interval ($I_\alpha = \{x / \mu_I(x) \geq \alpha\}$) is a classical interval. A decomposition by α-cuts can be used to compute the function on fuzzy intervals. ($[f(X_1, \cdots, X_n)]_\alpha = f([X_1]_\alpha, \cdots, [X_n]_\alpha)$). For example,

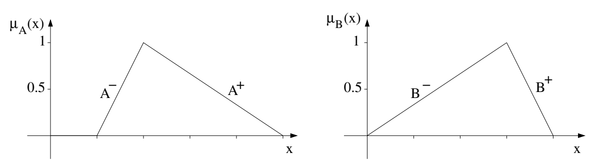

Fig. 1. Possibility distribution of two fuzzy intervals A and B

let A and B the fuzzy intervals on Figure 1. Let C be the maximum of A and B ($C = \widetilde{max}(A, B)$). C is defined at level α by $C_\alpha = \overline{max}(A_\alpha, B_\alpha)$, where $\overline{max}$ is the operator maximum on classical intervals ($\overline{max}([a, b], [c, d]) = [max(a, c), max(b, d)]$).

Now, we call left profile (this notion will be formally defined in the next Section) the increasing part of a fuzzy interval I (denoted I^-), and right profile its decreasing part (I^+). To obtain the left-profile (respectively right-profile) of C, we make a horizontal comparison, taking the maximum, of the left-profiles (respectively right-profiles) of A and B (Figure 2). Then C is completely defined by C^- and C^+ (Figure 3). Similarly from the same fuzzy intervals A and B,

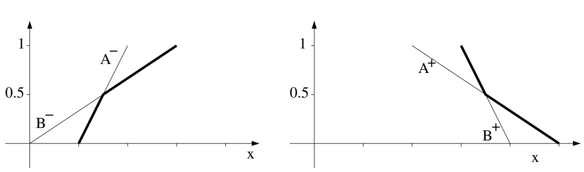

Fig. 2. Horizontal comparison of left profiles (on left) and right profiles (on right) of A and B

let us compute the difference $D = A \ominus B$. To obtain the left profile (respectively right profile) of D, we can compute

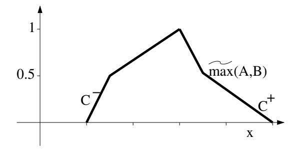

Fig. 3. Maximum of A and B

the horizontal difference of the left profile of A and the right profile of B (respectively the right profile of A and the left profile of B) (see Figure 4). In the first example

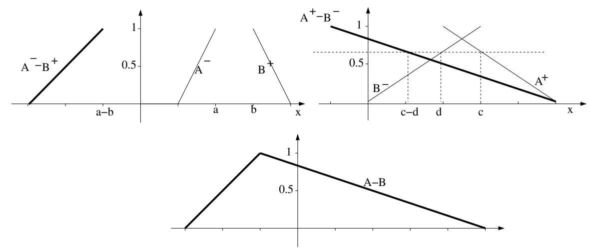

Fig. 4. Difference $A - B$

($C = \widetilde{max}(A, B)$), we have obtained the left profile of C from both left profiles because the function $h(x, y) = max(x, y)$ is increasing with respect to both x and y. We have computed the right profile of C with the right profiles of A and B for the same reason. The function $g(x, y) = x - y$ being increasing in x and decreasing in y, we get the left profile of D subtracting B^+ from A^-, and the right profile D^+ subtracting A^+ from B^-.

Let us now formalise this intuitive approach rigorously.

IV. Profiles And Fuzzy Configurations

We need to define an object (called profile) to handle the increasing or decreasing part of a fuzzy interval, and also operations between such objects.

Definition 5: A *profile* is a function Φ from $[0, 1]$ to $\mathbb{R}$.

Note that a profile is not requested to be monotonic. In the following, $[s^-, s^+]$ will represent the support of I.

Definition 6: Let I be an USC fuzzy interval. We call *left profile of* I (denoted I^-) the profile defined as follows:

$$\begin{array}{llll} I^- : & [0,1] & \longrightarrow & \mathbb{R} \\ & \lambda & \longmapsto & I^-(\lambda) = inf\{x | \mu_I(x) \geq \lambda, \; x \geq s^-\} \end{array}$$

We call *right profile of* I (denoted I^+) the profile defined as following:

$$\begin{array}{llll} I^+ : & [0,1] & \longrightarrow & \mathbb{R} \\ & \lambda & \longmapsto & I^+(\lambda) = sup\{x | \mu_I(x) \geq \lambda, x \leq s^+\} \end{array}$$

This definition seems complex, but it permits to preserve USC properties across computations, and is simple to use in practice. With this definition, an USC fuzzy interval can be entirely defined by its left profile and its right profile (this is not the case for all fuzzy sets, since two profiles only define a convex membership function): From I, we can obtain I^+ and I^- by construction (see Definition 6), and conversely, given I^+ and I^-, we derive μ_I as follow:

Theorem 1: Let I be a USC fuzzy interval, I^- and I^+ its left and right profiles, then the membership function μ_I of I

is defined in the following way:

$$\mu_I(x) = \begin{cases} 0 & if\ x < I^-(0) \\ sup\{\lambda | I^-(\lambda) \leq x\} & if\ I^-(0) \leq x < I^-(1) \\ 1 & if\ I^-(1) \geq x \geq I^+(1) \\ sup\{\lambda | I^+(\lambda) \geq x\} & if\ I^+(1) < x \leq I^+(0) \\ 0 & if\ I^+(0) < x \end{cases}$$

Now, let us give the definition of a fuzzy extreme configuration.

Definition 7: Let $x = (x_1, x_2, \cdots, x_n)$ a tuple of n independent variables, restricted by the fuzzy intervals $X_1, \cdots, X_n$. A *fuzzy extreme configuration* Ω is a n-tuple of left or right profiles: $\Omega = (X_1^{\epsilon_1}, X_2^{\epsilon_2}, \cdots, X_n^{\epsilon_n})$, where $\epsilon_i \in \{+, -\}$. We denote $\widetilde{\mathcal{H}}$ the set of all fuzzy extreme configurations: $\widetilde{\mathcal{H}} = \times_i \{X_i^-, X_i^+\}$ ($|\widetilde{\mathcal{H}}| = 2^n$)

We denote Ω_i the i^{th} profile of configuration Ω. For any $\Omega \in \widetilde{\mathcal{H}}$, let $\Omega(\lambda)$ denote the classical configuration obtained at level λ. $\Omega(\lambda) = (\Omega_1(\lambda), \Omega_2(\lambda), \cdots, \Omega_n(\lambda)) \in \mathbb{R}^n$ is a vertex of the hyper-rectangle $\times_i [X_i]_\lambda$.

The definition of a fuzzy interval as a pair of profiles is akin to the so-called graded numbers of Herencia [6]. This author also considers fuzzy numbers as mappings from the unit interval to the set of real intervals, instead of the usual USC mapping from the reals to the unit interval. However, profiles are more general because they are not necessarily monotonic. Only monotonic profiles are useful to define fuzzy intervals, however, as shown in the sequel, computations with fuzzy intervals may lead to non-monotonic profiles as intermediary results. That is why profiles are defined as functions from $[0,1]$ to $\mathbb{R}$, which associates for each possibility level $\lambda \in [0,1]$ a single abscissa $\Phi(\lambda)$. For example the result of the maximum of A^+ and B^- (A and B defined by Figure 1) is not a function from $\mathbb{R}$ to $[0,1]$ (see Figure 5).

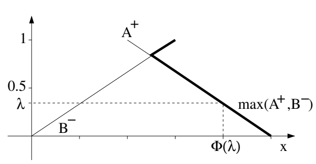

Fig. 5. Maximum of A^+ and B^-

Non-monotonic profiles can thus appear in the intermidiate computations, but hopefully, the final result is always a classical fuzzy interval. The above example is meant for illustration of non-monotonic profiles.

If the membership function of an upper semi-continuous fuzzy interval has some discontinuity points, it can be useful to display its profiles on separate graphs. See the profiles of I (Figure 6) respect Definition 6, and therefore, the membership function of I can be exactly recovered from I^- and I^+ according to Theorem 1. Note that the left and right profiles of an USC fuzzy interval are both left-continuous.

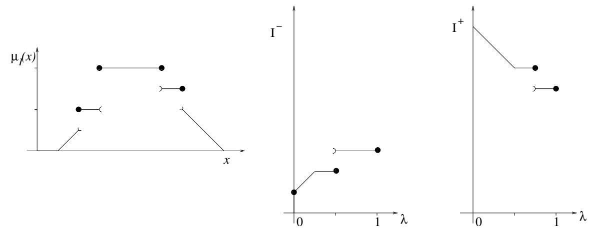

Fig. 6. Examples of profiles of a non continuous fuzzy interval

V. Calculation And Fuzzy Extreme Configurations

A. Main Results

We have defined notions of profiles and configurations for fuzzy variables. We can see now how to use these concepts for our purpose, which is to provide some tools for computing the range of a function under fuzzy interval arguments.

Definition 8: Let f be a function of arity n. Let us denote $\dot{f}$ the extension of f applicable to profiles: for any n-tuple of profiles $\Omega = (\Omega_1, \Omega_2, \cdots, \Omega_n)$, $\dot{f}(\Omega)$ is the profile defined as follows: $\forall \lambda \in [0,1]$

$$\begin{aligned} \dot{f}(\Omega)(\lambda) &= f(\Omega(\lambda)) \\ &= f(\Omega_1(\lambda), \Omega_2(\lambda), \cdots, \Omega_n(\lambda)) \end{aligned}$$

For instance, the extension of maximum and subtraction has been used to obtain the profiles of Figures 2, 4 and 5.

Now, let us define a set $\xi \subseteq \{-,+\}^n$ such that for all intervals $\mathcal{X} = \times_i [x_i^-, x_i^+]$, ξ defines a set of configurations $\mathcal{H}_{\mathcal{X},\xi}$ as follows: $\mathcal{H}_{\mathcal{X},\xi} = \{(x_1^{\epsilon_1}, \cdots, x_n^{\epsilon_n}) | (\epsilon_1, \cdots, \epsilon_n) \in \xi\}$ If there are n fuzzy intervals $X_1, \cdots, X_n$, ξ also defines a set of fuzzy configurations: $\widetilde{\mathcal{H}}_\xi = \{(X_1^{\epsilon_1}, \cdots, X_n^{\epsilon_n}) | (\epsilon_1, \cdots, \epsilon_n) \in \xi\}$. With these notations we can state the following theorem:

Theorem 2: Let $x = (x_1, x_2, \cdots, x_n)$ be a tuple of n independent variables, restricted by the fuzzy intervals $X_1, \cdots, X_n$, defined by their membership functions $\mu_{X_1}, \cdots \mu_{X_n}$ all USC. f is a function from $\mathbb{R}^n$ to $\mathbb{R}$, and Y is the fuzzy set of the possible values of the variable $y = f(x)$. If there is a set $\xi \subseteq \{(\epsilon_1, \cdots, \epsilon_n), \epsilon_i \in \{-,+\}\}$, such that for all α-cuts f attains its maximum and minimum on $\mathcal{X}_\alpha = \times_i [X_i]_\alpha$ for a configuration in $\mathcal{H}_{\mathcal{X}_\alpha,\xi}$,
then $Y^+ = \dot{max}_{\Omega \in \widetilde{\mathcal{H}}_\xi} \{\dot{f}(\Omega)\}$
and $Y^- = \dot{min}_{\Omega \in \widetilde{\mathcal{H}}_\xi} \{\dot{f}(\Omega)\}$

Proof: Let $\lambda \in [0,1]$ be a possibility degree. By definition of the right profile, we know that:

$$\begin{aligned} Y^+(\lambda) &= max\{y | \mu_Y(y) \geq Y^\lambda\} \\ &= max\{y | y = f(x_1, \cdots, x_n), x_i \in X_i^\lambda\} \end{aligned}$$

And then, under the hypothesis of the theorem, we can write that: $Y^+(\lambda) = max\{y | y = f(x_1^{\epsilon_1}, \cdots, x_n^{\epsilon_n}), (\epsilon_1, \cdots, \epsilon_n) \in \xi\}$,
which exactly means $Y^+(\lambda) = \dot{max}_{\Omega \in \widetilde{\mathcal{H}}_\xi} (\dot{f}(\Omega)(\lambda))$
This equation is true for all $\lambda \in [0,1]$, therefore, we can conclude that $Y^+ = \dot{max}_{\Omega \in \widetilde{\mathcal{H}}_\xi} \{\dot{f}(\Omega)\}$ □

As in the interval case, we can state two corollaries based on the monotony of f:

Corollary 1: Under the assumption of Theorem 2, if f is locally monotonic with respect to each argument,

then $Y^- = \dot{min}_{\Omega \in \widetilde{\mathcal{H}}}(\dot{f}(\Omega))$
and $Y^+ = \dot{max}_{\Omega \in \widetilde{\mathcal{H}}}(\dot{f}(\Omega))$

Proof: This is Theorem 2, where $\xi = \{(\epsilon_1, \cdots, \epsilon_n), \epsilon_i \in \{-, +\}\}$ $\square$

Corollary 2: Under the assumption of Theorem 2, if f is locally monotonic with respect to each argument, and $\forall j \in E_1$, f is increasing according to x_j and $\forall j \in E_2$, f is decreasing according to x_j, then

$$Y^- = \dot{min}_{\Omega \in \widetilde{\mathcal{H}}}\left(\dot{f}(\Omega) \middle| \begin{array}{l} \forall j \in E_1, \Omega_j = X_j^- \\ \forall j \in E_2, \Omega_j = X_j^+ \end{array}\right)$$

and $Y^+ = \dot{max}_{\Omega \in \widetilde{\mathcal{H}}}\left(\dot{f}(\Omega) \middle| \begin{array}{l} \forall j \in E_1, \Omega_j = X_j^+ \\ \forall j \in E_2, \Omega_j = X_j^- \end{array}\right)$

Proof: Obvious with Theorem 2 and Definition 6 of profiles. $\square$

This last corollary was in fact known for strictly increasing functions [5]. In the remainder of this paper, we will not tell f from its extension applicable to the profiles $\dot{f}$.

B. Why This Approach Is Useful In Practice

The above theory looks complex, but lots of applications are really simple. Let us imagine some computations on piecewise linear fuzzy intervals (such fuzzy sets are not hard to implement [7] [8]). The profiles of such fuzzy sets are obviously piecewise linear and can be implemented in the same way. Some operations on such profiles preserve the piecewise linearity property: for example the maximum, minimum, addition, subtraction. Moreover, for addition and subtraction, no new break-points are generated, and for the minimum or maximum, the number of break-points may double in the worst case. Then we can deduce that the complexity of a computation on piecewise linear profiles with these operators is polynomial according to the number of break-points. The implementation of our computation method on functions defined from these operations is easy and can be generalized to other operations.

C. Example Of Computation Using Profiles

Let us see a simple example of application of Corollary 1. Let h be the function defined by $h(x, y) = max(x, y) - y$. h is locally monotonic with respect to each argument. Let a and b be two independent variables with values in the fuzzy intervals A and B, defined by the membership functions of Figure 7.

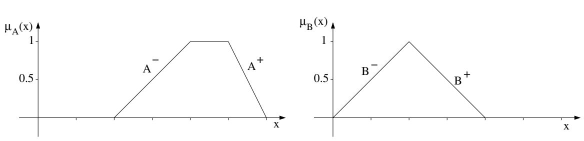

Fig. 7. Membership function of the fuzzy intervals A and B

We denote by c the variable defined by $c = h(a, b)$. Now, we can compute the fuzzy set C of possible values of c from the expression where y appears twice.

The set of fuzzy extreme configurations is
$\widetilde{\mathcal{H}}_1 = \{(A^-, B^-), (A^+, B^-), (A^+, B^-), (A^+, B^+)\}$.
According to Corollary 1, we can apply h on each element of $\widetilde{\mathcal{H}}_1$ (Figure 8), put all the results of these computations on the same graph, and compute their fuzzy convex hull (Figure 9). Note that we get a non-monotonic profile on Figure 8 as

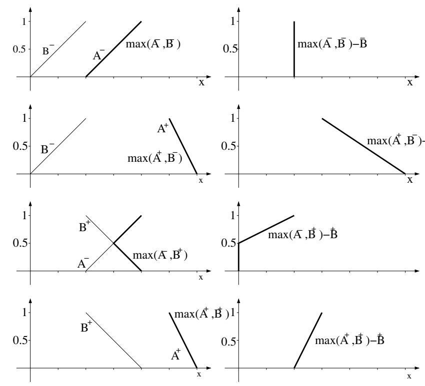

Fig. 8. Details of the computation of h applied on $\hat{H}_1$

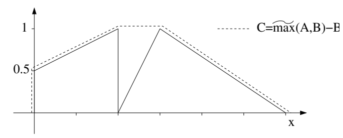

Fig. 9. Superposition of the result of h applied on $\hat{H}_1$

partial result.
In fact, the function h is increasing according to x, and decreasing according to y. This is obvious since: $h(x, y) = max(x - y, 0)$. If we had noticed it earlier, the computation would have been easier: Corollary 2 recommends to use only configuration on $\widetilde{\mathcal{H}}_2 = (A^+, B^-), (A^+, B^-)$. Therefore only the second and the third line of the Figure 8 would have been useful to determine C completely. However, it is not always possible to rewrite the function in such a way that each variable appears once only.

VI. Application To Standard Problems

The previous computation method can be easily applied to more difficult problems. Our motivation first came from the scheduling problem, but during our work on this domain, we saw other problems to which this theory can be applied.

A. Multiplication

A very simple application of this method can be the multiplication of two fuzzy numbers overlapping 0. For this application, we can take two fuzzy interval given by their L-R parametrized representations [5]. Let L be any USC from $[0, +\infty)$ to $[0, 1]$. satisfying the following requirements: $\forall x > 0,\ L(x) < 1$; $\forall x < 1,\ L(x) > 0$; $L(0) = 1$; either $L(1) = 0$ or ($\forall x,\ L(x) > 0$, and $lim_{x \to +\infty} L(x) = 0$. Under these requirements, L is said to be a shape function. Two shape functions L and R and a Four-tuple $(\underline{m}, \overline{m}, s, t)$ such that $\underline{m}$, $\overline{m} \in \mathbb{R} \cup \{-\infty, +\infty\}$, s, $t \in [0, +\infty)$, define a fuzzy interval A by the following equation:

$$\begin{array}{l} \mu_A(x) = L(\frac{\underline{m} - x}{s}),\ \forall x < \underline{m} \\ \mu_A(x) = 1,\ \forall x \in [\underline{m}, \overline{m}] \\ \mu_A(x) = R(\frac{x - \overline{m}}{t}),\ \forall x > \overline{m} \end{array} \quad (1)$$

The left profile of A is the function A^- defined by $A^-(\lambda) = \underline{m} - s * L^{-1}(\lambda)$, and the right-profile A^+ of A is the function $A^+(\lambda) = t * R^{-1}(\lambda) + \overline{m}$.

The multiplication of two L-R parametrized fuzzy intervals A and B defined by their four-tuple $(\underline{m}, \overline{m}, s, t)$ and $(\underline{n}, \overline{n}, u, v)$ can then be done by Corollary 1. Indeed, The function $h(x, y) = x * y$ is locally monotonic on $\mathbb{R}^2$. Then we can conclude that the following equations are valid:
$(A * B)^- = min(A^- * B^-, A^+ * B^-, A^- * B^+, A^+ * B^+)$
$(A * B)^+ = max(A^- * B^-, A^+ * B^-, A^- * B^+, A^+ * B^+)$
It extends a well-known formula of interval arithmetics [1] (page 12). Note that for two profiles Φ, Ψ, $min(\Phi, \Psi) \neq \Phi$ and $min(\Phi, \Psi) \neq \Psi$ (see for exemple Figure 5). This computation is easy with the usual tools in the case of non-negative fuzzy interval, but our result can be applied to all fuzzy intervals. For example, consider the two fuzzy intervals A and B, defined by Figure 10. A (respectively B) is a L-L

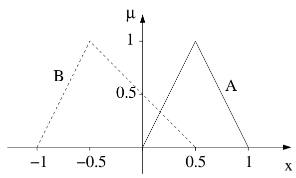

Fig. 10. Possibility distribution of two fuzzy intervals A and B

parametrized fuzzy interval for $L(x) = 1 - x \ \forall x \in [0, 1]$ and $L(x) = 0 \ \forall x > 1$, and the four-tuple $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (respectively $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 1)$). Now, $L^{-1} = L$, therefore the profiles of A and B are defined as follows:
$A^-(\lambda) = \frac{\lambda}{2}$, $A^+(\lambda) = 1 - \frac{\lambda}{2}$,
$B^-(\lambda) = \frac{\lambda}{2} - 1$, $B^+(\lambda) = \frac{1}{2} - \lambda$
Then we get: $(A^- * B^-)(\lambda) = \frac{\lambda}{2} * (\frac{\lambda}{2} - 1)$
$(A^+ * B^-)(\lambda) = (1 - \frac{\lambda}{2}) * (\frac{\lambda}{2} - 1)$
$(A^- * B^+)(\lambda) = \frac{\lambda}{2} * (\frac{1}{2} - \lambda)$
$(A^- * B^-)(\lambda) = (1 - \frac{\lambda}{2}) * (\frac{1}{2} - \lambda)$

The computed profile and the result $C = A * B$ are shown on figure 11. The above calculations are in the style of graded numbers [6] but some profiles obtained as partial results are not monotonic.

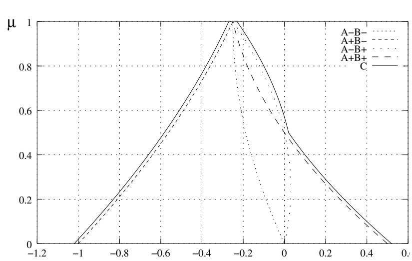

Fig. 11. $C = A * B$ and the computed profiles

B. Scheduling Problem

For a general description of scheduling problems, the reader should refer to [9].

1) Problem Definition: A scheduling problem can be defined by a set of tasks (or activities) which represents the different parts of a project, and a set of precedence constraints expressing that some tasks cannot start before others are completed. In this context, the goal of a project manager is generally to minimize the makespan of the project. Three quantities are computed for each task of the project (they allow to identify the critical tasks): the *earliest starting date* e_i of a task i is the date before which we cannot start the task without violation of a precedence constraint. The *latest starting date* l_i is the date after which we cannot start the task without delaying the end of the project. The *float* f_i is the difference between the latest starting date and the earliest starting date. A task is then critical iff its float is null.

These three quantities are computed by the PERT Algorithm based on three equations which only use min, max, $+$ and $-$ operators:

$$\begin{aligned} e_i &= max\{W(p) | p \in P_{1,i}\} \\ &= max\{e_j + d_j | j \in pred(i)\} \\ l_i &= max\{W(p) | p \in P_{1,n}\} - max\{W(p) | p \in P_{i,n}\} \\ &= min\{l_j - d_j | j \in succ(i)\} \\ f_i &= l_i - e_i \end{aligned}$$

where d_j is the duration of the task j, $pred(i)$ is the set of tasks preceding i, $succ(i)$ is the set of tasks following i. We note $P_{i,j}$ the set of all path from task i to task j, and $W(p_{i,j})$ the length of path $p_{i,j} \in P_{i,j}$.

e_i is the length of the longest path from the starting task (noted 1) to task i. l_i is the length of the longest path from the starting task to the ending task (noted n) minus the longest path from task i to the ending task.

In *scheduling problems under uncertainty* (on fuzzy PERT) a task duration can be modeled by an interval, crisp or fuzzy.

2) Application Of The Profile Method: The expressions of e_i and l_i obey monotony properties and a set of configurations can be pointed out where the bounds of the quantities are attained [10] [11]. From these results and the theorems of the last section, it is now easy to work on the fuzzy version of the problem.

D_i is the fuzzy interval representing the possible valuations of the duration of task i. E_i L_i and F_i are respectively the fuzzy earlier starting date, latest starting date, and float of i.

The problem defines for each task i three functions computing the earliest starting dates $e_i(.)$, the latest starting dates $l_i(.)$ and the floats $f_i(.)$. These functions of n variables (if there are n tasks in the problem) take a n-tuple of task durations as parameter.

First $e_i(.)$ is increasing according to each argument. So applying Corollary 2, we directly get two expressions that compute the fuzzy earlier starting date:
$E_i^- = max\{E_j^- + D_j^- | j \in pred(i)\}$
$E_i^+ = max\{E_j^+ + D_j^+ | j \in pred(i)\}$
Similarly, for the fuzzy latest starting date L_i and float F_i, we know that the functions $l_i(.)$ and $f_i(.)$ are locally monotonic with respect to each argument. Moreover, we can find a subset of variables according to which $l_i(.)$ and $f_i(.)$ are increasing [10]: $l_i(x_1, \cdots, x_n)$ is increasing with respect

to all x_j such that $j \notin succ(i) \cup \{i\}$, and $f_i(x_1, \cdots, x_n)$ is increasing with respect to all x_j such that $i \notin pred(i) \cup \{i\} \cup succ(i)$. Therefore we can apply Corollary 2. No more configurations are necessary in the fuzzy case than in the interval case.

From the results in [10], we have develloped another algorithm (the Path-Algorithm) for the interval-valued problem [12], in which, the computations of $l_i(.)$ and $f_i(.)$ can be done on a *small* set of configurations. This set is the basis of the application of Theorem 2 in the fuzzy version of the problem. The minimum of $l_i(.)$ is attained on a configuration $\omega = (x_1^{\epsilon_1}, \cdots, x_n^{\epsilon_n})$ where the set of task assigned to their maximum ($\{i|\epsilon_i = +\}$) is exactly a path from task i to the endding task n. The maximum of $l_i(.)$ and $f_i(.)$ and the minimum of $f_i(.)$ is attained on a configuration $\omega = (x_1^{\epsilon_1}, \cdots, x_n^{\epsilon_n})$ where the set of tasks assigned to their maximum ($\{i|\epsilon_i = +\}$) is exactly a path from the starting task 1 to the endding task n. With Theorem 2, we obtain the exact fuzzy profiles of the latest starting dates and floats with the same time complexity as in the crisp case.

C. Fuzzy Weighted Average

The fuzzy weighted average problem is: given n fuzzy weights W_i and n fuzzy X_i, how to obtain the fuzzy weighted average of the variable $y = fwa(w_1, \cdots, w_n, x_1, \cdots, x_n) = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}$.

The (FWA) Algorithm [3] decomposes the problem into M interval problems (corresponding to M α-cuts). Then for each α-cut, it computes $fwa(w_1, \cdots, w_n, x_1, \cdots, x_n)$ on each vertex of the hyper-rectangle $(W_1)_\alpha \times \ldots (W_n)_\alpha \times (X_1)_\alpha \times (X_n)_\alpha$, where $(Z)_\alpha$ is the α-cut of Z at level α. The maximal (respectively minimal) possible value of z at possibility level α is then the greatest (respectively the lowest) computed value. This is due the the fact that the function $fwa(.)$ is locally monotonic with respect to each argument:

Proposition 3: The function $fwa(.)$ is locally monotonic with respect to each argument (according to Definition 3)

Applying Corollary 1 to the fuzzy weighted average yields a generalized (FWA) Algorithm, which gives an exact value of the average, for each possibility level, with a time complexity in $O(2^n)$. On the contrary, the classical (FWA) Algorithm gives the exact value of the average only for a restricted number (say M) of possibility degrees (the rest of the result is approximated) with complexity $O(M * 2^n)$. Note that polynomial algorithms have been recently developed for the real interval problem [13], but these algorithms can not be extended to the present profile theory. Their extension is left to further research.

In the case where all x_i are precisely defined, we can order the x_i such that for all $j < i$, $x_j \leq x_i$. With this order, there exists $k \in [1, n-1]$ such that for all $i \leq k$, $fwa(.)$ is decreasing with respect to w_i, and for all $i > k$, $fwa(.)$ is increasing with respect to w_i. So we can apply Theorem 2 with the set $\xi = \{(-, +, \cdots, +), (-, -, +, \cdots, +), \cdots, (-, \cdots, -, +)\}$, and so we obtain a linear algorithm to compute the fuzzy weighted average $Z = fwa(W, x)$, extending to profiles the one of Lee and Park [14].

VII. Conclusions

We have designed a new approach for fuzzy computations problem. We have seen on several problems under uncertainty how our method can be applied. This list is of course not exhaustive and lots of problems should find answers with this computation method.

We have shown that for locally monotonic functions, interval analysis techniques are easily extended to fuzzy intervals using profiles. In particular it can be applied to arithmetic operations which are not monotonic on the whole real line (multiplication, division), to FWA and scheduling.

The profile method was developed for functions which reach their maximum and minimum values on the bounds of interval entries. In basic problems of interval computation, this is of course not always the case. For example with a differentiable function of $(x_1, \cdots, x_n)$, for each variable x_i extrema of f for $x_i \in [x_i^-, x_i^+]$ reached on x_i^-, or x_i^+, or can be on a point $x_i^\sim$ where partial derivatives of f equal to 0. So it seems possible to model the third possibility in the case of fuzzy problem by adding a constant profile $X_i^\sim(\lambda) = x_i^\sim$ for all λ. But this is a topic for further research.

References

[1] R. Moore, *Methods and applications of interval analysis*. SIAM Studies in Applied Mathematics, 1979.

[2] H. Q. Yang, H. Yao, and D. Jones, "Calculating functions of fuzzy numbers," *Fuzzy Set and Systems*, vol. 55, pp. 273–283, 1993.

[3] W. Dong and F. Wong, "Fuzzy weighted average and implementation of the extension principe," *Fuzzy Set and Systems*, vol. 21, pp. 183–199, 1987.

[4] J. Buckley, "Fuzzy pert," in *Applications of fuzzy set methodologies in industrial engineering*. Elsevier, 1989, pp. 103–114.

[5] D. Dubois, E. Kerre, R. Mesiar, and H. Prade, "Fuzzy interval analysis," in *Fundamentals of Fuzzy Sets*. Kluwer, 2000, pp. 483–581.

[6] J. Herencia and M. Lamata, "A total order for the graded numbers used in decision problems," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 7, pp. 267–276, 1999.

[7] E. Kerre and R. Baekeland, "Piecewise linear fuzzy quantities: a way to implement fuzzy information into expert systems and fuzzy databases," in *Uncertainty and Intelligent Systems*. Springer-Verlag, 1988, pp. 119–126.

[8] E. Kerre, H. Steyaert, F. V. Parys, and R. Baekeland, "Implementation of piecewise linear fuzzy quantities," *International Journal of Intelligent Systems*, vol. 10, pp. 1049–1059, 1995.

[9] R. Bellman, A. Esogbue, and I. Nabeshima, *Mathematical aspects of scheduling & applications*. Pergamon Press, 1982.

[10] D. Dubois, H. Fargier, and V. Galvagnon, "On latest starting times and floats in activity networks with ill-known durations," *European Journal of Operation Research*, vol. 147, pp. 266–280, 2003.

[11] D. Dubois, H. Fargier, and P. Fortemps, "Fuzzy scheduling: modeling flexible constraints vs. coping with incomplete knowledge," *European Journal of Operation Research*, vol. 147, pp. 231–252, 2003.

[12] D. Dubois, H. Fargier, and J. Fortin, "Computational methods for determining the latest starting times and floats of tasks in interval-valued activity networks," submitted to Journal of Intelligent Manufacturing.

[13] Y.-Y. Guh, C.-C. Hon, K.-M. Wang, and E. S. Lee, "Fuzzy weighted average: a max-min paired elimination method," *Computers Math. Applic.*, vol. 32, pp. 115–123, 1996.

[14] D. Lee and D. Park, "An efficient algorithm for fuzzy weighted average," *Fuzzy Set and Systems*, vol. 87, pp. 39–45, 1997.