Characterizing Strongly Normalizing Terms of a λ-Calculus with Generalized Applications via Intersection Types

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ΛJ — **Motivation.**

Cut-free derivations in Gentzen's **LJ**.

$$\frac{}{\Gamma, A \vdash A} \qquad \frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}(L\to) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

With term annotations. Read $x(s, z.t)$ as: let $z = xs$ in t.

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma \vdash s : A \qquad \Gamma, z : B \vdash t : C}{\Gamma, x : A \to B \vdash x(s, z.t) : C}(L\to) \qquad \frac{\Gamma, x : A \vdash r : B}{\Gamma \vdash \lambda xr : A \to B}$$

Closure under substitution — without types.

$$\Lambda\mathrm{J} \ni r, s, t ::= x \mid r(s, z.t) \mid \lambda xr.$$

(The closure of the computational content of cut-free LJ under substitution—hence the name.)
Embedding from Λ into ΛJ and vice versa.

$$rs := r(s, z.z) \in \Lambda\mathrm{J}, \qquad r(s, z.t) := t[z := rs] \in \Lambda.$$

ΛJ — Inductive characterization and normal forms.

Notation $R, S, T ::= (r, x.s)$.

Inductive characterizations.

$$
\begin{array}{lllllll}
\Lambda\mathrm{J} \ni r, s, t ::= x & \mid x(s, z.t) & \mid \lambda x r & \mid x R S \vec{S} & \mid (\lambda x r)(s, z.t) & \mid (\lambda x r)(s, z.t) S \vec{S} \\
\mathrm{NF}_\beta \ni r, s, t ::= x & \mid x(s, z.t) & \mid \lambda x r & \mid x R S \vec{S} & & \\
\mathrm{NF}_\pi \ni r, s, t ::= x & \mid x(s, z.t) & \mid \lambda x r & & \mid (\lambda x r)(s, z.t) & \\
\mathrm{NF} \ni r, s, t ::= x & \mid x(s, z.t) & \mid \lambda x r & & &
\end{array}
$$

We get back Gentzen's LJ!

$$(\lambda x r)(s, z.t) \mapsto_\beta t[z := r[x := s]]$$

$\mapsto_\beta$ is motivated by the embedding into Λ.

$$r(s, z.t)R \mapsto_\pi r \underbrace{(s, z.tR)}_{(s,z.t)\{R\}:=}$$

Both sides of $\mapsto_\pi$ have the same encoding in Λ.

Critical pairs.

$$\begin{tikzcd}
(\lambda x r)(s, z.t)R \arrow[r, mapsto, "\pi"'] \arrow[d, "\beta"] & (\lambda x r)(s, z.tR) \arrow[d, "\beta"] \\
t[z := r[x := s]]R \arrow[r, equal] & (tR)[z := r[x := s]]
\end{tikzcd}$$

$$\begin{tikzcd}
rRST \arrow[rr, mapsto, "\pi"'] \arrow[d, "\pi"] & & rRS\{T\} \arrow[d, mapsto, "\pi"] \\
rR\{S\}T \arrow[r, mapsto, "\pi"'] & rR\{S\}\{T\} \arrow[r, "\pi"'] & rR\{S\{T\}\}
\end{tikzcd}$$

Syntax-directed characterization of strongly normalizing terms.

Inductively define SN (which characterizes the strongly normalizing terms even when $r(s, z.t) \mapsto_\kappa t$ for z not free in t is added for the sake of the subformula property).

$$\frac{}{x \in \mathsf{SN}} \qquad \frac{s, t \in \mathsf{SN}}{x(s, z.t) \in \mathsf{SN}}\,(\mathrm{V})$$

$$\frac{r \in \mathsf{SN}}{\lambda x r \in \mathsf{SN}}\,(\lambda) \qquad \frac{x(s, z.tS)\vec{S} \in \mathsf{SN}}{x(s, z.t)S\vec{S} \in \mathsf{SN}}\,(\pi)$$

$$\frac{t[z := r[x := s]]\vec{S} \in \mathsf{SN} \qquad r, s \in \mathsf{SN}}{(\lambda x r)(s, z.t)\vec{S} \in \mathsf{SN}}\,(\beta)$$

Intersection type assignment.

Inductively define $\Gamma \vdash r : \rho$ with types ρ generated from identifiers by the binary connectives $\to$ and $\cap$.

$$\frac{}{\Gamma, x : \rho \vdash x : \rho}\,(\mathrm{V}) \qquad \frac{\Gamma, x : \rho \vdash r : \sigma}{\Gamma \vdash \lambda x r : \rho \to \sigma}\,(\lambda)$$

$$\frac{\Gamma \vdash r : \rho_1 \qquad \Gamma \vdash r : \rho_2}{\Gamma \vdash r : \rho_1 \cap \rho_2}\,(\cap\text{-I}) \qquad \frac{\Gamma \vdash r : \rho_1 \cap \rho_2 \qquad i \in \{1, 2\}}{\Gamma \vdash r : \rho_i}\,(\cap\text{-E})$$

$$\frac{\Gamma \vdash r : \rho \qquad \Gamma \vdash s : \rho_1 \qquad \Gamma, z : \rho_2 \vdash t : \tau \qquad z \in \mathrm{FV}(t) \Rightarrow \rho = \rho_1 \to \rho_2}{\Gamma \vdash r(s, z.t) : \tau}\,(\mathrm{app})$$

Remark on (app).

Rule (app) written differently:

$$\frac{\Gamma \vdash r : \rho_1 \to \rho_2 \qquad \Gamma \vdash s : \rho_1 \qquad \Gamma, z : \rho_2 \vdash t : \tau}{\Gamma \vdash r(s, z.t) : \tau}\,(\mathrm{app}_1)$$

AND

$$\frac{\Gamma \vdash r : \rho \qquad \Gamma \vdash s : \sigma \qquad \Gamma \vdash t : \tau \qquad z \notin \mathrm{FV}(t)}{\Gamma \vdash r(s, z.t) : \tau}\,(\mathrm{app}_2)$$

(app_1) is the rule we read off our presentation of Gentzen's LJ.
(app_2) is essential for completeness!

Important properties.

Substitution Lemma:

(a) If $\Gamma, x : \rho \vdash r : \tau$ and $\Gamma \vdash s : \rho$ then $\Gamma \vdash r[x := s] : \tau$ (usual substitution).

(b) If $\Gamma \vdash r[x := s] : \tau$, $x \notin \mathrm{FV}(s) \cup \Gamma$ and $(x \notin \mathrm{FV}(r) \Rightarrow \Gamma \vdash s : \rho_0)$ then there is a type ρ such that $\Gamma, x : \rho \vdash r : \tau$ and $\Gamma \vdash s : \rho$ (inverse substitution, typical of intersection types).

Inversion Lemma:

Let $\Gamma \vdash r : \rho$ with ρ prime type (= no intersection).

1. If $r = x$ then $x : \cap\vec{\rho} \in \Gamma$ with $\rho \in \vec{\rho}$.
2. If $r = \lambda xt$ then $\rho = \rho_1 \to \rho_2$ and $\Gamma, x : \rho_1 \vdash t : \rho_2$.
3. If $r = r'(s, z.t)$ then $\Gamma \vdash r' : \sigma$, $\Gamma \vdash s : \sigma_1$, $\Gamma, z : \sigma_2 \vdash t : \rho$ and $(z \in \mathrm{FV}(t) \Rightarrow \sigma = \sigma_1 \to \sigma_2)$.

Even easier than for Λ since $\cap\vec{\rho}$ only appears in clause 1 and not in the case of an application.

Subject Reduction: If $\Gamma \vdash r : \rho$ and $r \to r'$ then $\Gamma \vdash r' : \rho$.

Completeness.

We show that the set of typable terms has all the defininig closure properties of SN, hence every strongly normalizing term is typable.

First we consider the most delicate subcases. We even get preservation of types under those expansions (which does not hold below λ-abstractions but only for the head expansions shown).

Lemma (β-expansion). If $\Gamma \vdash t[z := r[x := s]] : \tau$, $\Gamma, x : \sigma' \vdash r : \rho$ and $\Gamma \vdash s : \sigma$ then $\Gamma \vdash (\lambda x r)(s, z.t) : \tau$.

Lemma (π-expansion). If $\Gamma \vdash r(s, z.tS) : \tau$ then $\Gamma \vdash r(s, z.t)S : \tau$.

Theorem (completeness). If $r \in$ SN then there are Γ and ρ such that $\Gamma \vdash r : \rho$.

Proof by induction on SN.

The lemmas have to be extended by $\vec{S}$ via the two parts of the substitution lemma.

Incompleteness without rule (app_2).

Lemma. Let $\vec{\rho}$ be prime types. The following is impossible:
(Note that the term $\lambda x.x(x, z.z)$ translates into $\omega := \lambda x.xx$ in Λ.)

$$\Gamma \vdash \lambda x.x(x, z.z) : \cap\vec{\rho} \to \sigma \text{ and } \Gamma \vdash \lambda x.x(x, z.z) : \cap\vec{\rho}.$$

Proof by induction on the number of different types in $\vec{\rho}$.

Corollary. $(\lambda x.x(x, z.z))(\lambda x.x(x, z.z), y.z) \in \mathsf{SN}$ but not typable in the restricted ($=$ without (app_2)) intersection typing system.

Remark: also follows from soundness. (If the above term were typable in the restricted system, then also with $y.y$ instead of $y.z$, and that term is the $\Lambda\mathrm{J}$-counterpart of $\omega\omega \in \Lambda$, well-known not to be normalizing—in contradiction with soundness.)

Soundness (= strong normalization).

Lemma (derived π-rule for SN). If $r(s, z.tS)\vec{S} \in$ SN then $r(s, z.t)S\vec{S} \in$ SN.

Lemma. For all types ρ, for all $r \in$ SN:

(i) if $\Gamma \vdash r : \rho$, $\Gamma \vdash s : \rho_1$, $\Gamma, z : \rho_2 \vdash t : \tau$ and $s, t \in$ SN and $(z \in \mathrm{FV}(t) \Rightarrow \rho = \rho_1 \to \rho_2)$ then $r(s, z.t) \in$ SN,

(ii) if $\Gamma, x : \rho \vdash r : \tau$, $\Gamma \vdash s : \rho$ and $s \in$ SN then $r[x := s] \in$ SN.

Remark: In case (i) we have $\Gamma \vdash r(s, z.t) : \tau$ and in case (ii) by the Substitution Lemma $\Gamma \vdash r[x := s] : \tau$.

The proof is by simultaneous induction on ρ, side induction on $r \in$ SN, and case analysis according to $r \in$ SN. One always first has to prove (i) and later infers (ii), possibly with the help of (i).

Main Theorem.

Theorem (Soundness). If $\Gamma \vdash r : \rho$ then $r \in$ SN.

Proof by induction on $\vdash$, using (i).

Hence, SN is the set of typable terms, and since SN is the set of strongly normalizing terms, we are done.

Remark: Another proof by saturated sets (computability predicates) is in the workshop proceedings.

Conclusions

By intersection types we learn more about ΛJ which is interesting in itself as the untyped core of a notation system for cut-free sequent calculus, closed under substitution.

This study is the first time that permutative conversions (which are an extensional concept) and intersection types have been brought together.

A normalization proof for intersection types in the style of a one-pass cut-elimination has been shown.

ΛJ is a minimal model for the study of the notoriously unpleasant permutative conversions (there are no junk terms as in λ-calculus extended by a case construct).

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