The computational content of Zorn's lemma

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Zorn's lemma is a well known formulation of the axiom of choice which states that any chain complete partially ordered set has a maximal element. Certain theorems in mathematics can be given a particularly elegant proof using Zorn's lemma - a well-known example of this is the theorem that any ring with unity has a maximal ideal. In this talk I will focus on giving a computational interpretation to Zorn's lemma. More precisely, I will describe a new form of recursion which realizes the functional interpretation of certain restricted instances of Zorn's lemma.

There are two main motivating factors behind this work. The task of making constructive sense of Zorn's lemma is an interesting and challenging proof theoretic problem in its own right. My emphasis here is on providing a *natural* realizer for the functional interpretation of the lemma which clearly reflects its computational content. This alone is a non-trivial task, as even in the weak cases of Zorn's lemma considered here such a realizer will necessarily be based on an extremely strong form of recursion, undefinable even in Gödel's system T. The second factor is that a direct computational interpretation of Zorn's lemma should enable us to extract intuitive programs from non-constructive proofs which rely on it. This in particular paves the way for a proof theoretic analysis of several important theorems in abstract algebra and well-quasi order theory that make use of choice in this form.

My talk builds on a number of recent studies which examine the constructive meaning of variants of Zorn's lemma, most importantly the work of U. Berger [1], who has given a direct and elegant modified realizability interpretation of a reformulation of the lemma known as open induction. The difference here is that I work in the alternative setting of Gödel's functional interpretation (which requires a different realizing term) and look towards giving a more general interpretation. Moreover, I emphasise the algorithmic behaviour of the realizer, linking it to my own recent research on giving learning-based realizers to the functional interpretation of classical principles [2]. The talk is very much about work in progress, and I aim to emphasise open problems and directions for future research.

References

[1] U. Berger. A computational interpretation of open induction. In *Proceedings of LICS 2004*, pages 326–334. IEEE Computer Society, 2004.

[2] T. Powell. Gödel's functional interpretation and the concept of learning. To appear in Proceedings of LICS 2016.