On the Limitations of Ordinal Approaches to Decision-Making

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Abstract

The aim of this paper is to evaluate the capabilities and the limitations of purely ordinal approaches to decision under uncertainty (DMU) and Multi-Criteria Decision Making (MCDM). Results obtained in this paper are valid for the two decision problems, due to their strong mathematical similarity. In the axiomatic approach followed by this paper, ordinality is captured by an axiom requiring that two ordinally equivalent pairs of alternatives should be ranked in the same way. We prove that this requirement is basically incompatible with the axiomatic framework that underlies most numerical approaches. Ordinality is especially incompatible with the transitivity of preference. When the requirement of transitivity is relaxed to quasi-transitivity, the resulting ordinal decision rules are more elaborate than what impossibility theorems in social choice suggest. Yet they prove to have a rather limited expressive power. Some ways out of this difficulty are suggested at the end of the paper.

1 INTRODUCTION

In the last ten years, decision-making has received much attention in artificial intelligence. Yet, it is a topic where scientific traditions exist, and have been around for a long time independently of AI. There are at least three decision making problems that have been studied in the past: decision making under uncertainty (DMU), social choice (SC) and multi-criteria decision making (MCDM).

In DMU, the most popular approach is numerical and relies on expected utility. This approach has been adopted in AI with the development of techniques such as influence diagrams and Markov decision processes. In social choice, the tradition which comes back to Arrow's impossibility theorem is basically ordinal. The situation is more diversified in MCDM, where both numerical approaches (multi-attribute utility theory) and ordinal approaches (outranking methods) coexist.

In artificial intelligence, the nineties have witnessed the emergence of a qualitative approach to decision making. This trend is motivated by knowledge representation concerns, namely the search for expressive languages capable of expressing preferences and reasoning with them. However, in such attempts, classical numerical decision rules are more or less given up and the question arises as to whether non numerical decision rules are rational at all and what is their expressive power.

The aim of this paper is to evaluate the capabilities and the limitations of purely ordinal decision criteria in DMU and MCDM. Results presented are valid for both decision problems, due to their mathematical similarity. We follow an axiomatic methodology, where ordinality is captured by an axiom requiring that two ordinally equivalent pairs of alternatives be ranked in the same way. We show that this requirement is basically incompatible with the axiomatic framework that underlies most numerical approaches and is especially incompatible with the transitivity of preference. In other words, this paper lays bare the wide gap existing between qualitative and numerical decision theories.

When the requirement of transitivity is relaxed to quasi-transitivity, the resulting ordinal decision rules are more elaborate than what impossibility theorems in social choice suggest. Yet they prove to have a rather limited expressive power. They are either too adventurous or not enough decisive. Some ways out of this difficulty are suggested : either by restoring some form of commensurateness forbidden by pure ordinality or, by introducing aspiration levels.

2 TWO DECISION PARADIGMS

This Section surveys the existing frameworks in decision under uncertainty and multi-criteria decision making and shows the formal similarity between them.

2.1 DECISION MAKING UNDER UNCERTAINTY

A decision-making problem under uncertainty is a 4-tuple $(N, X, \mathcal{A}, \succsim)$, where:

- N is the set of the possible *states of nature*,
- X is the set of the possible *consequences* of acts,
- $\mathcal{A} = X^N$ is the set of *potential acts*, that is, the set of functions $x : N \to X$,
- $\succsim$ is a *preference relation* on $\mathcal{A}$, usually reflexive and transitive.

Notation For any preference relation $\succsim$, its asymmetric part defines a strict preference denoted $\succ$ and defined by: $x \succ y \Leftrightarrow (x \succsim y$ and not$(y \succsim x))$; its symmetric part defines an indifference relation denoted $\sim$ and defined by $x \sim y \Leftrightarrow (x \succsim y$ and $y \succsim x)$.

The preference relation $\succsim$ on $\mathcal{A}$ usually relies on the use of a *preference functional*, i.e. a function that maps acts to an ordered value scale. More formally, a preference function f is said to represent a preference relation $\succsim$ if and only if:

$$x \succsim y \Leftrightarrow f(x) \geq f(y)$$

Most of the preference functionals involve a numerical function u on X encoding the utility or the relative attractiveness of the consequences and a measure μ from 2^S to $[0, 1]$ representing the likelihood of the events (seen as sets of states). The latter measure must be a *capacity* on N, i.e. a mapping μ from 2^S to $[0, 1]$ such that $\mu(\emptyset) = 0, \mu(N) = 1$, and $\forall A, B \subseteq N, A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.

For instance, the seminal work of Savage [28] advocates a numerical approach, whereby uncertainty is represented by a probability function, preference is encoded by a utility functions and acts are ranked according to a particular preference functional, namely expected utility (written here in the finite setting, although Savage assumes N is infinite):

$$eu(x) = \sum_{i \in N} p(i)u(x(i)) \quad (1)$$

Expected utility has been extended, among others, by Sarin and Wakker [27] on the basis of Choquet integral [29]. The Choquet integral is a powerful aggregation operator allowing positive and negative synergies between events by means of capacity functions other than classical probabilities (for instance, the likelihood of $A \cup B$ can be greater or less than the sum of the individual likelihoods of A and B).

Escaping the highly quantitative nature of these decision rules, more qualitative models have been proposed, that rely on ordinal evaluation scales. Among them, the pessimistic qualitative preference functional of [11, 12] is based on a utility function u from X to an ordinal scale and a possibility distribution π from S to the *same* scale. It is defined by the following preference functional :

$$v^-(x) = \min_{i \in N} \max\{1 - \pi(i), u(x(i))\} \quad (2)$$

This criterion is actually a *weighted* extension of the Wald maximin criterion. It is a particular case of a more general criterion based on a qualitative counterpart to the Choquet integral (the Sugeno integral [31, 12]).

Recent works in AI propose to escape the use of a preference functional combining utility degrees and plausibility degrees, and they do it in two different ways. The first approach is to compare two acts on the basis of their consequences in the *most plausible states of the world* [2, 32, 7, 8, 9, 33], using more or less complex criteria. This class of approaches also aims at providing the user not only with a decision rule, but also with a high-level language (typically, a logic) for the description of the preferences, e.g. a base of conditional preferences [2] or a base of defaults encoding either desires or beliefs [33]. The criteria used in these domains are generally qualitative, e.g. criteria such as maximin, minimax regret and competitive ratio axiomatized in [7, 8, 9]. The second approach is to rely on a decision rule that compares the plausibility of the sets of states in which one act has a better consequence than the other [10, 14]. These models, that generalize Condorcet's relative majority rule, are based on pairwise comparisons. They do not compute a preference functional for each act, but start by comparing, for any pair x, y in $\mathcal{A}$ and each state i in N, the consequences $x(i)$ and $y(i)$ in order to decide whether x is at least as good as y according to this state. Then, one prefers act x to act y if and only if it is more likely to get a better consequence with x than with y. Formally, the rules of pairwise comparison (or "concordance rules") are of the form :

$$x \succsim y \Leftrightarrow C_{\succsim}(x, y) \succsim_\Lambda C_{\succsim}(y, x) \quad (3)$$

where $C_{\succsim}(x, y) = \{i \in N, x(i) \succsim_P y(i)\}$, $\succsim_P$ is a preference relation on X, and $\succsim_\Lambda$ is a likelihood relation on 2^N. Relation $\succsim_P$ is generally a weak order on consequences (e.g. the one encoded by a utility function). Relation $\succsim_\Lambda$ can be derived from a plausibility function (i.e., a capacity μ) if any (then $A \succsim_\Lambda B \Leftrightarrow \mu(A) \geq \mu(B)$) or introduced as such by the DM. One only requires that $\succsim_\Lambda$ be a monotonic set-relation:

Definition 1 (Monotonic set-relation) *A set-relation on S is a relation $\succsim_\Lambda$ on 2^S such that $\succsim_\Lambda$ is reflexive, $\succ_\Lambda$ is transitive, $S \succ_\Lambda \emptyset$, $\forall A \subseteq N, (S \succsim_\Lambda A$ and $A \succsim_\Lambda \emptyset)$.*

$\succsim_\Lambda$ *is said to be* monotonic *if and only if:*
$\forall A, B, A', B'$ *such that* $A \subseteq A'$ *and* $B' \subseteq B$,
$A \succsim_\Lambda B \Rightarrow A' \succsim_\Lambda B'$.

$\succsim_\Lambda$ *is said to be* preadditive *if and only if:*
$\forall A, B, C$ *such that* $A \cap (B \cup C) = \emptyset$,
$B \succsim_\Lambda C \Leftrightarrow A \cup B \succsim_\Lambda A \cup C$.

Monotonic set-relations extend the basic properties of capacities to (possibly, partial) relations : any capacity obviously induces a monotonic set-relation: $A \succsim_\Lambda B \Leftrightarrow \mu(A) \geq \mu(B)$. Moreover, the monotonic set-relations satisfy the fundamental property of capacities ($A \subseteq B \implies A \succsim_\Lambda B$).

2.2 MULTICRITERIA DECISION-MAKING

A multi-criteria decision-making problem is again a 4-tuple $(N, \mathcal{A}, \mathcal{X} \succsim)$, where :

- $N = \{1, \ldots, n\}$ is the set of *attributes* or *criteria* describing the alternatives,
- $\mathcal{X} = \{X_1, \ldots, X_n\}$ is the set of domains of the n attributes - X_j denotes the set of possible values for attribute $j \in N$,
- $\mathcal{A}$ is the Cartesian product $\Pi_{i \in N} X_i$: it is the set of *alternatives*,
- $\succsim$ is a *preference relation* on $\mathcal{A}$.

MCDM is of the same structure as DMU, the set of criteria playing the role of states and the alternatives playing the role of acts. The main difference between MCDM and DMU is that MCDM attaches a domain X_i to each $i \in N$, although in DMU all the states $i \in N$ take their values in the same X. In other terms, a problem of DMU can be understood as a MCDM problem where $X_1 = \cdots = X_n = X$. Besides, N is always finite in MCDM. Elements of X_j are denoted x_j, and the alternative x stands for the vector $(x_1, \ldots, x_n)$.

Interestingly, there are here two distinct schools of thought in MCDM, one deriving from the DMU tradition, the other one from voting theory. The first school (see eg [20]) is essentially numerical and puts forward the aggregation of marginal utility functions $u_j(x_j)$ (measuring the attractiveness of the attribute value x_j) into a global preference functional $u(x)$ - notice that in contrast with DMU, MCDM does not use a unique utility measure; it rather uses as many marginal utilities as elements of N. One principally finds here the weighted sum, which is a counterpart to Savage's expected utility, and its generalization by the Choquet integral [18, 23]. In a more qualitative framework, one should prefer the Sugeno integral (e.g. a weighted minimum, similar to pessimistic utility). Finally, decision rules also based on aggregation functions have been proposed [24] that do not correspond to weak orders of alternatives, but can encompass weaker notions such as semi-orders, or interval orders and involve indifference thresholds; namely, the classical representation $x \succsim y \Leftrightarrow u(x) \geq u(y)$ is extended into $x \succsim y \Leftrightarrow \phi(u(x), u(y)) \geq 0$, where ϕ is a comparison function, e.g. $\phi(u(x), u(y)) = u(x) - u(y) + a$, where a is a (positive) tolerance threshold for the (weak) preference of x over y.

The second school stems from the works of Roy [25]: preferences along each criteria (whether numerical or not) are represented by an outranking relation. So-called additive concordance rules are used to perform criteria aggregation: they are based on counting the number of criteria which favor an alternative over another. This kind of rule is a particular case of a general concordance rule similar to the rule of pairwise comparison introduced in the previous Section. This general rule defines a preference relation $\succsim$ on X from the marginal preference relations $\succsim_j$ on X_j, $\forall j = 1, \ldots, n$ and a relation $\succsim_\Lambda$ on 2^N as per equation 3 where $C_\succsim(x, y) = \{j \in N, x_j \succsim_j y_j\}$ is now the set of criteria where x is as least as good as y and $\succsim_\Lambda$ is a monotone set-relation that encodes the relative importance of the coalitions of criteria. Again, $\succsim_\Lambda$ can be derived from an importance function or introduced as such from scratch by the decision maker.

3 ORDINALITY

In order to capture the essence of pure ordinality in DMU and MCDM we will introduce an axiom of *Ordinal Invariance*, expressing that, on each component j (attribute or state), only the relative positions of the values (of the consequences) of the two alternatives (of the two acts) are important. In the following, N is supposed to be finite, as it is the case in MCDM.

Let us first introduce the notion of compound act (or compound alternative): for any pair $x, y \in \mathcal{A}$ and for any subset of components (states or criteria) $A \subseteq N$, we can construct a mixed act/alternative xAy whose components are those of x on the elements of A, and those of y on the elements of $N \setminus A$. Formally :

$$\forall i \in N, \ (xAy)(i) = \begin{cases} x(i) & \text{if } i \in A, \\ y(i) & \text{if } i \notin A \end{cases}$$

For instance, starting from the initial relation $\succsim$, the relative position of two values on a component j is expressed by the marginal relations $\succsim_j$ defined as follows:

Definition 2 $\forall x, y \in \mathcal{A}, \forall j \in N$, $(x_j \succsim_j y_j \Leftrightarrow \forall z \in \mathcal{A}$, $(x\{j\}z) \succsim (y\{j\}z))$

There are components that never make any difference, i.e. such that $x_j \succsim_j y_j, \forall x_j, y_j \in X_j$. They are said to be *null*. More generally, one can express the conditioning of $\succsim$ on any subset $A \subseteq N$ and thus define null subsets as follows :

Definition 3 $\forall A \subseteq N, \forall x, y \in \mathcal{A} : (x \succsim y)_A \Leftrightarrow \forall z \in \mathcal{A}, xAz \succsim yAz$

Definition 4 *$A \subseteq N$ is said to be* null*if and only if* $\forall\, x,\, y \in \mathcal{A}, (x \succsim y)_A$

$\succsim$ also induces a relation on the subsets of N, interpreted as the relative likelihood of events when N is a set of states, or as the relative importance of coalitions of criteria when N is a set of criteria:

Definition 5 *$\forall A, B \subseteq N$, $A \succsim_L B$ if and only if $(xAy) \succsim (xBy)$ holds for any $x, y \in X$ such that, for all non-null $\{j\} \subseteq N$, $x_j \succ_j y_j$.*

This definition is very natural: suppose that for all x and y such that x is uniformly better than y, xAy is preferred to xBy. The reason for this state of facts is obviously that the event A is more likely than the event B (or, in MCDM, that the coalition of attributes A is considered as more important than the coalition B).

Let us now state an axiom of ordinality first proposed in [14] expressing that only the relative positions of the values of the two alternatives are important, but not the values themselves, nor the positions of the two alternatives relatively to other alternatives : it means that changing the values of x and y on any component, in such a way that the order between the components is preserved, cannot change the relative preference between x and y. In other words, two ordinally equivalent pairs of alternatives should be ranked in the same way.

Definition 6 *Two pairs of alternatives (x, y) and (z, w) are said to be ordinally equivalent, denoted $(x, y) \equiv (z, w)$ if and only if :*$[\forall j \in N, (x_j \succsim_j y_j \Leftrightarrow z_j \succsim_j w_j)$ *and* $(y_j \succsim_j x_j \Leftrightarrow w_j \succsim_j z_j)]$

Axiom OI $\forall x, y, z, w \in \mathcal{A} : (x, y) \equiv (z, w) \Rightarrow (x \succsim y \Leftrightarrow z \succsim w)$.

This axiom is an adaptation to the case of weak preference relations of the non-compensation condition used in [16, 3]. It can be seen as the counterpart to an independence condition used in social choice theory [30], completed with a neutrality condition making preferences independent of the labels of the alternatives.

It is obvious that the pairwise comparison rule presented in Section 2 satisfy OI. The ordering of alternatives or acts induced by such rules is not affected by any monotonic transformation of the value scales of marginal utility functions that encode the preference according to each component, nor by any monotonic transformation of the scale encoding the importance of criteria (or, in DMU, the likelihood of events). This is why we may claim that ordinality goes along with the *non-commensurateness* of the distinct notions involved, namely the non-commensurateness of uncertainty and utility in DMU, and, in MCDM, the non-commensurateness of criteria importance and marginal utilities (and, in addition, the non-commensurateness of the different marginal scales).

Actually, axiom OI is generally not satisfied by the approaches based on a preference functional, e.g. expected utility, qualitative utility nor their counterparts in MCDM. Let us see in more details why this is so, by embedding OI into the axiomatics of Savage, since the latter justifies the use of expected utility.

4 ORDINALITY IN DMU

First, let us simplify our definitions from the fact that, in DMU, the same set of consequences is considered for each state. A first consequence of this restriction is that, among acts in $\mathcal{A}$ there are *constant acts* such that: $\exists\, a \in X : \forall\, i \in N,\, x(i) = a$. Such an act will be denoted f_a.

In Savage's framework, it seems reasonable to identify the set of constant acts and X. In this case, all the marginal preference relations $\succsim_j$ between elements of X are supposed to correspond to the global preference $\succsim_P$ derived from constant acts, provided that j is not null. This is expressed by the third axiom of Savage:

Axiom P3 $\forall A \subseteq N$, *A not null*, $\forall a, b \in X$, $\forall z \in \mathcal{A}$, $f_a A z \succsim f_b A z \Leftrightarrow a \succsim_P b$.

The DM preference relation $\succsim_P$ on X can then be soundly induced from $\succsim$ by:

$$\forall a, b \in X,\ \ (a \succsim_P b \Leftrightarrow f_a \succsim f_b) \qquad (4)$$

Assuming P3, Savage then defines the relative likelihood of events given by:

Definition 7 $A \succsim_L B \Leftrightarrow$ $(\forall a, b \in X$ *such that* $a \succ_P b$, $(f_a A f_b) \succsim (f_a B f_b))$

This definition is in general not totally equivalent to definition 5 (it gives a finer $\succsim_L$). However, both definitions are equivalent when P3 and OI are assumed. Hence, relation $\succsim_L$ can be simply derived from comparisons involving only pairwise combinations of constant acts. In this context, it can be shown that OI acts as a strong version of the second and fourth axioms of Savage. Indeed, P2 (the sure thing principle) expresses that, if x and y share the same consequence a_j on some state j (or coincide on some set of states J), then changing this (these) consequence(s) into another one (into others) will not change the global preference. OI encompasses this case, since it says that changing $x(j)$ and $y(j)$ into any pair of consequences while preserving the same preference relation as $x(j)$ and $y(j)$ (and not only the indifference induced by the equality) does not change the global preference. Formally:

Axiom P2 (Sure-Thing Principle). $\forall A \subseteq N$, $\forall x, y, z, z' \in \mathcal{A}, (xAz \succsim yAz \Leftrightarrow xAz' \succsim yAz')$

Proposition 1 *(OI + $\succsim$ reflexive) $\Rightarrow$ P2*

Moreover, the fourth Savage axiom P4 is simply an application of OI to binary acts (i.e. two acts of the form $f_a A f_b$, where $a \succ_P b$).

Axiom P4 $\forall A, B \in \mathcal{A}, \forall a, b, a', b' \in X$ such that $a \succ_P b$ and $a' \succ_P b'$, $aAb \succsim aBb \Leftrightarrow a'Ab' \succsim a'Bb'$.

Proposition 2 *(OI + $\succsim$ reflexive) $\Rightarrow$ P4*

So, ordinality is highly compatible with P2, P3, P4, and obviously with the fifth axiom of Savage (that says that $\exists a, b \in X$ such that $a \succ_P b$) or its natural strengthening P5′, that says that at least three non-indifferent consequences exist :

Axiom P5′ $\exists a, b, c \in X$ such that $a \succ_P b$ and $b \succ_P c$.

Major difficulties arise when P1, the first axiom of Savage, is added, which requires that $\succsim$ be a weak order :

Axiom P1 $\succsim$ is complete and transitive.

Theorem 1 *If $\succsim$ is a preference relation that satisfies P1, P3, P5′ and OI , then :*

$\forall i, j \in N, \{i\} \sim_L \{j\} \Leftrightarrow$ *i and j are null.*

$$\forall x, y \in \mathcal{A} : x \succ y \Leftrightarrow \exists i^* \text{ s. t. : } x(i^*) \succ_P y(i^*) \text{ and } \forall j \succ_L i^*, x(j) \sim_P y(j)$$

Corollary 1 *If $\succsim$ is a preference relation that satisfies P1, P3, P5′ and OI , then :*

$$\exists i^* \in N : \begin{cases} \{i^*\} \succ_L N \setminus \{i^*\} \\ \forall x, y \in \mathcal{A},\ x(i^*) \succ_P y(i^*) \implies x \succ y \end{cases}$$

This theorem shows the lexicographic structure of the preference induced by P1 in a purely ordinal context and can be seen as a counterpart to Fishburn's theorem obtained in the context of multicriteria analysis [15] (similarly, Corollary 1 is a counterpart to Arrow's Theorem [1] in social choice). The resulting rule corresponds to a very particular structure of uncertainty in which all the non impossible states must be linearly ordered. Moreover, there must exist a predominant state i^*, more likely that any other state, and even more plausible than the event $N \setminus \{i^*\}$ - i.e. P1 can be satisfied if and only if one has the intuition to be in a quasi-deterministic situation. As soon as i^* gives a better consequence for x than for y, the first act is preferred. The other possible states of the world are so negligible compared to i^* that they are not considered. They only might be influent in the comparison of particular pairs (x, y) such that $x \sim_{i^*} y$.

So, ordinality is in conflict with P1. In Savage's work, P1 is justified by the goal assigned to the theory. If acts are ranked according to expected utility then the preference is reflexive, transitive and complete. But the DM Preferences could be rational without being complete nor transitive. More precisely, the reflexivity of $\succsim$ is obviously not questionable, nor the transitivity of the strict preference. But it makes sense to leave room for incomparability, apart from strict preference and indifference, so as to allow for a better description of indecisive situations: indifference between two acts x, y can be restricted to the rough similarity between the two acts whereas incomparability reflects the existence of a conflict in the comparison of x and y. So, $\sim$ is not necessarily transitive and $\succsim$ is not necessarily complete. On the other hand, it is obvious that there is no conflict between states in the comparison of constant acts. So, we may solely require the total comparability (i.e. the completeness) of constant acts. Hence we relax P1 into a weaker axiom A1:

Axiom A1 $\succsim$ is reflexive, quasi-transitive (i.e. $\succ$ is transitive) and its restriction to constant acts is complete.

However, when weakening P1, we lose important consequences of {P1, P2, P3, P4, P5}, like for instance the monotony of $\succsim$:

Definition 8 *Let $\succsim$ be a preference relation on $\mathcal{A}$ such that $a \succ_P b$ and $b \succsim_P c \Rightarrow a \succsim_P c$. $\succsim$ is said to be monotonic if and only if the two following conditions hold:*

Left Monotony (LM)*:*
if $\forall i \in A \subseteq N, x'(i) \succ_P x(i)$, *then* $x \succsim y \Rightarrow x'Ax \succsim y$

Right Monotony (RM)*:*
if $\forall i \in A \subseteq N, y(i) \succ_P y'(i)$, *then* $x \succsim y \Rightarrow x \succsim y'Ay$

This condition (denoted MO in the sequel) requires that, if x is preferred to y, then locally enhancing x or degrading y will obviously not reverse the preference. This condition obviously makes sense only in the context of a preference on consequences satisfying $a \succ_P b$ and $b \succsim_P c$ imply $a \succsim_P c$, e.g. when $\succsim_P$ is complete and quasi-transitive - this is the case when A1 holds . In Savage's framework, the latter property follows from P1 and moreover :

Proposition 3 *If $\succsim$ satisfies P1, P2 and P3 then $\succsim$ is monotonic.*

If P1 is not accepted a priori, this monotonicity property should be explicitly added to the set of axioms so as to retain as much as possible from the framework of Savage

Let us now introduce the main result of this Section, that states that ordinality not only rules out the approaches based on a preference functional (because OI is almost incompatible with P1), but it enforces the use of a rule of pairwise comparison for representing the preference on acts:

Theorem 2 *If $\succsim$ is complete on constant acts, reflexive, satisfies P3 and OI then there exists a preference relation $\succsim_P$ on X and a likelihood relation $\succsim_L$ on 2^N such that:*

$$\forall\, x, y \in \mathcal{A} : x \succsim y \Leftrightarrow C_{\succsim_P}(x, y) \succsim_L C_{\succsim_P}(y, x)$$

where $C_{\succsim_P}(x, y) = \{i \in N, x(i) \succsim_P y(i)\}$.

$\succsim_P$ is obviously the comparison of consequences derived from the comparison of constant acts by equation (4), and $\succsim_L$ is the likelihood relation given by equation (7). Let us focus on the latter relation. The main feature of $\succsim_L$ appears when $\succsim$ is monotonic, which is a very natural requirement, and when more than two distinct consequences exist, i.e. when P5$'$ holds:

Proposition 4 *If $\succsim$ is complete on constant acts, reflexive, verifies P3 and OI then:*

- *$\succsim_L$ is pre-additive,*
- *If $\succsim$ is monotonic, then $\succsim_L$ is a monotonic set-relation,*
- *If $\succsim$ is quasi-transitive and satisfies P5$'$, then $\succ_L$ satisfies the following property of* **Acceptance***:*
  $\forall A, B, C$ such that $A \cap B = A \cap C = B \cap C = \emptyset$,
  $A \cup C \succ_L B$ and $A \cup B \succ_L C \implies A \succ_L B \cup C$.

The first property comes from the validity of the Sure Thing Principle induced by OI. The last property is a very drastic one and enforces the qualitative nature of the likelihood relation. Some important remarks are in order:

- The property of Acceptance (or "Qualitativeness") has been previously identified in [13, 17, 19] and is characteristic of the conditional likelihood relations that are compatible with deductive closure. Namely, the set of accepted beliefs A in context C such that $A \cap C \succ_L (S \setminus A) \cap C$, induced by the relation $\succ_L$, is deductively closed. In particular, this property is almost incompatible with a probabilistic representation of uncertainty [10].
- It can be shown, following [10], that any monotonic set-relation that satisfies Acceptance underlies a consequence relationship that satisfies the properties of Kraus, Lehmann and Magidor's preferential entailment [21], but for the reflexivity (the latter must be restricted to non empty events: here, nothing can be deduced from contradictory contexts). In some sense, our system of postulates is a operationally testable axiomatization of System P in terms of DMU, like Savage's axioms are an act-driven justification of Probability Theory.
- The intersection of all the events A such that $A \succ_L N \setminus A$ forms the set of most plausible states - it can be shown that is is not empty. It is the one on the basis of which almost every decision is made: when comparing x and y, the states outside this set are not taken into account, unless each of the most plausible states is indifferent among the two acts ($x(i) \sim_P y(i)$). This makes such a framework very close to the approaches that make the decision on the basis of the most plausible states only - with the slight difference that, here, the less plausible states can be considered for breaking ties.
- The above remark holds when restricting to any context $C \subseteq N$, and the approach remain valid in a dynamic setting, when new information about the actual state of the word is acquired by the DM. Decisions are dynamically consistent.

5 ORDINALITY IN MCDM

Contrary to DMU, MCDM approaches have considered ordinality as a natural assumption early in their development, and the strong opposition existing between the requirement of $\succsim$ being a weak order and ordinality (expressed by the so-called axiom of non-compensation) has been stressed years ago, by Fishburn [15]. More generally, it has been shown in [35] that the mere requirement of quasi-transitivity of $\succsim$, when added to ordinality, enforces the existence of a unique predominant set of criteria called "Oligarchy".

Definition 9 *A set $O \subseteq N$ is said to be* predominant *if and only if it is decisive in N ($\forall x, y \in \mathcal{A}$: ($\forall i \in O, x_i \succ_i y_i$) $\implies$ $x \succ y$) and contains only vetoers (where i is a vetoer iff $\forall x, y \in \mathcal{A} : x_i \succ_i y_i \implies not(y \succ x)$).*

So, many researchers in MCDM tend to forget about non compensation, and thus focus on the study of multi-attribute decision-making using a numerical preference functional. The other school continued to stick to an ordinal approach to a large extent, through the study of pairwise comparison rules [34, 26, 4]. But, paradoxically, there are actually very few foundational works in ordinal MCDM, in comparison to multi-criteria numerical utility theory. One can however mention the pioneering work of Fishburn [16] (see also [3, 6, 5]). The results of the previous section suggest an axiomatic framework for pairwise comparison rules, mainly deriving from OI. However, the theorem cannot be directly translated from the framework of Section 4 to MCDM, since the latter context is more general, in the sense that the domains X_i (and the related marginal comparisons $\succsim_i$) can be different from one component to the other : axiom P3 does not make sense in MCDM. Similarly, the completeness on constant acts is not meaningful as such, but should be replaced by the the minimal comparability of values within each attribute domain X_i, expressing that no conflict exists when x and y differ on a single attribute only.

Axiom MC : $\forall j \in N, \forall x, y, z, \in X : (x\{j\}z) \succsim (y\{j\}z)$ or $(y\{j\}z) \succsim (x\{j\}z)$.

MC implies the reflexivity of $\succsim$ and the completeness of the $\succsim_j$. Moreover, it is easy to conclude from MC that: $\forall j \in N, j$ not null $\Rightarrow \exists x, y \in X, x_j \succ_j y_i$.

So, we can build two alternatives x^* and x_* such that $x^*_j \succ_j x_{*j}$ if j is not null and $x^*_j = x_{*j}$ if j is null. By OI, it is easy to show that the relative importance of coalitions of criteria (Definition 5), can be equivalently defined from x^* and x_*:

Proposition 5 *if OI holds, then* $A \succsim_L B \Leftrightarrow (x^*Ax_*) \succsim (x^*Bx_*)$

We can now derive an MCDM counterpart to Theorem 2:

Theorem 3 *If the decision maker's preference $\succsim$ satisfies OI and MC then there exists n complete preference relations $\succsim_1, \ldots, \succsim_n$, defined on $X_1, \ldots, X_n$ respectively, and a* a pre-additive *set-relation $\succsim_L$ on 2^N, such that:*

$$\forall x, y \in X, \;\; x \succsim y \;\Leftrightarrow\; C_{\succsim}(x, y) \succsim_L C_{\succsim}(y, x)$$

where $C_{\succsim}(x, y) = \{j \in N, x_j \succsim_j y_j\}$

It shows that, in MCDM too, pure ordinality enforces the use of a concordance rule. Moreover, we can adapt the results pertaining to the likelihood relation in DMU, to the importance relation induced on sets of criteria. For instance, if $\succsim$ is monotonic, then $\succsim_L$ is a monotonic set-relation. More importantly, when the strict part of $\succsim$ is transitive and satisfies a discrimination axiom that expresses P5$'$ in the context of MCDM, then $\succ_L$ again satisfies the property of Acceptance:

Axiom DI : $\forall j \in N, \exists x, y, z \in X$ such that $x_j \succ_j y_j$ and $y_j \succ_j z_j$.

Proposition 6 *If $\succsim$ is quasi-transitive and satisfies OI, MC and DI, then $\succ_L$ satisfies the property of Acceptance.*

It is well-known in MCDM and social choice that ordinality and quasi transitivity enforce the existence of a predominant set, i.e. of an oligarchy of vetoers. This is a consequence of the property of acceptance: the oligarchy is the intersection of all the A such as $A \succ_L N \setminus A$.

Theorem 4 *If $\succsim$ satisfies A1, MC, DI, MO and OI, then there exists one and only one coalition of predominant criteria.*

A similar result is known in Social Choice Theory [35] where such preference structures are called *oligarchic*. The larger the oligarchy, the less decisive the procedure. When all criteria have equal importance, the preference relation $\succsim$ reduces to the Pareto ordering. In the other extreme case, when the oligarchy contains a single criterion, the DM preferences can be described by a "dictatorial" decision rule.

In fact, the decision rule does not reduce to the existence of a mere oligarchy. Indeed, we know that the property of Acceptance, and thus the existence of a predominant event, is stable under conditioning.

Theorem 5 *If $\succsim$ satisfies A1, MC, DI, MO and OI, then, for any non-null $N' \subseteq N$, there exists a unique coalition O predominant in N', i.e. such that:*

- $O \subseteq N'$

- O is decisive in *N' :*
$\forall x, y \in \mathcal{A} : (\forall i \in O, x(i) \succ_i y(i)) \Rightarrow (x \succ y)_{N'}$,

- $\forall i \in O$, i is a vetoer in *N':*
$\forall x, y \in \mathcal{A} : x(i) \succ_i y(i) \Rightarrow$ *not*$((y \succ x)_{N'})$.

Moreover, O is such that: $\forall A \subseteq N' : O \subseteq A \Leftrightarrow A \succ_L N' \setminus A$

The thrust of these result is that when two alternative share the same attribute values on a subset E of attribute, these attributes are neglected (this is a consequence of the sure thing principle), and *there is a set of predominant criteria in $N \setminus E$ that acts as a local oligarchy* and can discriminate between the alternatives. This implies that there exists not only one oligarchy of criteria, but also a *hierarchy* of oligarchies, a result that had never been emphasized before:

Corollary 2 *If $\succsim$ satisfies A1, MC, DI, MO and OI, then there exists a partition of N into coalitions $O_1, \ldots, O_k, O_{k+1}$ such that:*

$$\forall j \leq k, O_j \textit{ is predominant in } N_j = N \setminus \bigcup_{i=1}^{j-1} O_i$$

Hence, whenever the first oligarchy does not make any difference between two alternatives, then the second level of the hierarchy becomes decisive and so on. This new result not only applies to MCDM, but also to the simpler framework of DMU. It means that, whenever the set of most plausible states is indifferent between two acts, then the decision rule considers the second level of plausibility (i.e. the most plausible states in $N \setminus O_1$) and so on.

6 CONCLUSION: THE WAY OUT

Regarding the question "Is decision-making possible in a purely ordinal framework?", the results presented in this paper provide a contrasted answer:

- Our axiomatic study shows that yes, if the invariance under ordinal transformations of the value scale, the uncertainty and the importance scales is accepted as a norm for qualitative models, then the only available decision rules are pairwise comparison rules and a minimal preference transitivity requirement is enough to almost rule out probabilistic (resp. additive) representations of uncertainty in DMU (resp. of criteria importance in MCDM). The uncertainty structures laid bare in DMU are precisely those at work in non-monotonic reasoning. The existence of an oligarchy of most plausible states makes this model close to the qualitative models where the decision is made on the basis of the most plausible states only [2, 32, 7, 8, 9, 33], even if these methods do not necessarily rely on the unanimity on these states. The existence of sub-oligarchies shows that qualitative decision making is not restricted to the single use of the most plausible states: less plausible states do play a role, when two acts are indifferent on the basis of the top oligarchy.

- On the other hand, the identification of hierarchies of oligarchies show that, no, this kind of approach is not decisive enough, unless one is in a situation of quasi-certainty (of quasi-dictatorship): as soon as the ranking of consequences of two acts (resp. alternatives) on two maximally plausible states (resp. maximally important criteria) are conflicting, the acts (resp. alternatives) are incomparable. Hence, the larger the oligarchies, the less decisive the rule. But as the rule becomes more decisive, it becomes more dictatorial. While in MCDM, deciding on the most important criteria is a very common attitude, it is more debatable in DMU as it comes down to neglecting less likely, yet not impossible states of nature, where consequences may be unpleasant. The resulting decisions become adventurous. Consider for instance the omelette example of Savage (see [28] pages 13 to 15). The problem is how to make a six-egg omelette from a five-egg one. The new egg can be fresh of rotten. There are three feasible acts : break the egg in the omelette (*bo*); break it apart in a cup (*bc*); or throw it away (*ta*). The set of 5 consequences is given in Table 1 . Integers between parentheses indicate the ordering of consequences (the greater the number the higher the preference). It is easy to check that $C_{\succ}(bo,bc) = C_{\succ}(bo,ta) = C_{\succ}(bc,ta) = \{\text{fresh}\}$ $C_{\succsim}(bc,bo) = C_{\succsim}(ta,bo) = C_{\succsim}(ta,bc) = \{\text{rotten}\}$. If fresh egg is more likely than rotten egg then the best act is clearly bo. If the decision-maker thinks the egg is rotten then the best act is ta. In case of total ignorance, the three acts are indifferent. So the decision making attitude induced by the approach is: break the egg in the omelette if you think the egg is fresh, throw it away if you think it is rotten, and do anything you like if you have no opinion . Clearly, this may result in many starving days, and garbage cans filled with lots of wasted fresh eggs, in the case when the state of the egg is usually hard to tell.

	fresh egg	rotten egg
bo	a 6 egg omelette (5)	nothing to eat (1)
bc	6 egg omelette + a cup to wash (4)	a 5 egg omelette + a cup to wash (2)
ta	5 egg omelette (3)	5 egg omelette (3)

Table 1: Savage's omelette example

So, if pure ordinal decision making is debatable except in very particular situations, is it possible to alter our initial framework in order to get more useful decision rules? A first idea is to relax quasi-transitivity into *acyclicity*. However, adapting results obtained in Social Choice Theory [30], it can easily be shown that non-compensatory decision models compatible with acyclicity then have very special features. For example, some state (resp. criterion) is necessarily an absolute vetoer, thus limiting the role of other states (resp. attributes).

Besides, it is clear that pairwise comparison rules are not the ultimate answer to qualitative decision problems. One may take into account the attitude of the DM in front of uncertainty like pessimism in the maximin criterion and their extensions. However such extensions require again commensurateness assumptions, even if a form of ordinality is preserved (as in [12], where the same ordinal scale is used for both uncertainty and utility).

One may also imagine situations where the preference between x and y depends on a third alternative z. As an example, consider the following relations:

$$x \Delta_z^+ y \quad \Leftrightarrow \quad C_{\succsim}(x,z) \succsim_L C_{\succsim}(y,z)$$
$$x \Delta_z^- y \quad \Leftrightarrow \quad C_{\succsim}(z,y) \succsim_L C_{\succsim}(z,x)$$

where $z \in \mathcal{A}$ is a reference point used to specify the aspiration levels of the DM, for each state of the nature. The idea underlying these rules is to compare acts by evaluating their relative capabilities of meeting the DM aspirations or not. Another interesting proposal would be to require $x\Delta_z^+ y$ and/or $x\Delta_z^- y$ for several alternatives z in a set Z. In MCDM, this set represents typical preference profiles. In DMU, choosing $Z = \{f_x, x \in X\}$ (constant acts) yields a qualitative counterpart to the so-called *stochastic dominance* (see e.g. [22]). These proposals preserve the ordinal nature of decision evaluations, and the non-commensurateness of the involved value scales. What they give up is an independence property contained in OI, which requires that $\succsim$ be context-independent, thus prescribing that the preference pattern between two alternatives or acts does not depend on their relative positions with respect to a third one. Decision rules that relax this independence assumption deserve further investigation.

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Appendix

Proof of Proposition 1. Consider $A \subseteq N$, $x,y,z,z' \in \mathcal{A}$. Since $\succsim$ is reflexive, so is also $\succsim_P$. It is then easy to check that $\forall i \in N, yAz(i) \succsim_P xAz(i) \Leftrightarrow yAz'(i) \succsim_P xAz'(i)$ and $\forall i \in N, xAz(i) \succsim_P yAz(i) \Leftrightarrow xAz'(i) \succsim_P yAz'(i)$. Hence $(xAz, yAz) \equiv (xAz', yAz')$ which implies $(xAz \succsim yAz \Leftrightarrow xAz' \succsim yAz')$ by OI.

Proof of Proposition 2. Let us choose arbitrary $A, B \subseteq N$ and $a, b, a', b' \in X$ such that $a \succ_P b$ and $a' \succ_P b'$. Since $\succsim$ is reflexive, so is also $\succsim_P$.
It is then easy to check that, $\forall i \in (A \cap B) \cup (\bar{A} \cap \bar{B})$: $aAb(i) \succsim_P aBb(i)$ and $a'Ab'(i) \succsim_P a'Bb'(i)$.
Moreover, $\forall i \in A \cap \bar{B}$: $aAb(i) = a$, $aBb(i) = b$, $a'Ab'(i) = a'$, $a'Bb'(i) = b'$. Since $a \succ_P b$ and $a' \succ_P b'$, we get $aAb(i) \succ_P aBb(i)$ and $a'Ab'(i) \succ_P a'Bb'(i)$.
Symmetrically, one can also check that for any $\forall i \in \bar{A} \cap B$: $aBb(i) \succ_P aAb(i)$ and $a'Bb'(i) \succ_P a'Ab'(i)$.
So, $(aAb, aBb) \equiv (a'Ab', a'Bb')$. Hence, using OI we get: $(aAb \succsim aAb \Leftrightarrow a'Ab' \succsim a'Bb')$ which establishes P4.

Proof of Theorem 1. It can be proved from Theorem 2 and Proposition 4 established later in this appendix. Indeed, they can be applied since P1, P3, P5$'$ and OI are assumed. Theorem 2 and the preadditivity property of $\succsim_L$ then implies that: $\forall x, y \in \mathcal{A}$, $x \succsim y \Leftrightarrow C_{\succsim}(x,y) \succsim_L C_{\succsim}(y,x) \Leftrightarrow C_{\succ}(x,y) \succsim_L C_{\succ}(y,x)$.

Two null events are obviously equiplausible. Reciprocally, suppose that $\exists i, j$ such that $\{i\} \sim_L \{j\}$ and $\{i\}$ not null. Consider a, b, c such that $a \succ_P b \succ_P c$ - they exist due to P5$'$ and P1 - and let us build the acts $x = b\{i\}c\{j\}c$, $y = c\{i\}a\{j\}c$ and $z = a\{i\}b\{j\}c$. $C_{\succ}(x,y) = \{i\}$, $C_{\succ}(y,x) = \{j\}$, $C_{\succ}(y,z) = \{j\}$, $C_{\succ}(z,y) = \{i\}$, $C_{\succ}(x,z) = \emptyset$, $C_{\succ}(z,x) = \{i,j\}$. $\{i\}$ not null implies $\{i,j\} \succ_L \emptyset$ thanks to proposition 4 and therefore $z \succ x$. On the other hand, $\{i\} \sim_L \{j\}$ implies $x \sim y$ and $y \sim z$, and thus, by P1 $x \sim z$, which contradicts $z \succ x$. So, $\{i\} \sim_L \{j\}$ implies $\{i\}$ null.

So, $N \setminus \{i,\ i \text{ null}\}$ forms a chain of singletons linearly ordered by $\succ_L$. We also know, by proposition 4 that $\succsim_L$ satisfies the property of Acceptance. It is easy to check that these two properties imply the following: $A \succ_L B \Leftrightarrow \exists i \in A$ such that $\{i\} \succ_L B$. Consider now a pair $(x,y) \in \mathcal{A}$ such that $x \succ y$. Since $x \succ y \Leftrightarrow C_{\succ}(x,y) \succ_L C_{\succ}(y,x)$, $x \succ y \Rightarrow \exists i \in C_{\succ}(x,y)$ such that $\{i\} \succ_L C_{\succ}(y,x)$. Let i^* be the most likely of these i: any state j more plausible than i^* is necessarily outside $C_{\succ}(x,y) \cup C_{\succ}(y,x)$. Hence, by completeness of $\succsim_P$, j belongs to $C_{\sim}(x,y)$.

Reciprocally, suppose that there exists an i^* such that $x(i^*) \succ_P y(i^*)$ and $x(j) \sim_P y(j)$ for all j such that $\{j\} \succ_L \{i^*\}$. Since the states are linearly ordered, i^* is more likely that any state of $C_{\succ}(y,x)$. So, using the property of acceptance, we get $C_{\succ}(x,y) \succ_L C_{\succ}(y,x)$, i.e. $x \succ y$.

Proof of Corollary 1 (sketch). Direct from Theorem 1, i^* being the maximally plausible state (it is unique since the states are linearly ordered). The property of acceptance implies that $i^* \succ_L S \setminus \{i^*\}$. $x(i^*) \succ_P y(i^*) \Rightarrow x \succ y$ is then deduced from $x \succsim y \Leftrightarrow C_{\succ}(x,y) \succsim_L C_{\succ}(y,x)$.

Proof of Proposition 3. We first establish the following:

Lemma 3.1 (Weak Unanimity) $\forall x, y \in \mathcal{A}, \forall A, B \subseteq N$ such that $A \cap B = \emptyset$, $((x \succsim y)_A$ and $(x \succsim y)_B) \Rightarrow (x \succsim y)_{A \cup B}$

Proof Consider x, y, A, B such that $A \cap B = \emptyset$ and that $\forall z$: $xAz \succsim yAz$ and $xBz \succsim yBz$. $P2$ allows us to write $xAxBz \succsim yAxBz$ and $yAxBz \succsim yAyBz$. By transitivity, this implies: $xAxBz \succsim yAyBz$ for any z, i.e. $(x \succsim y)_{A \cup B}$.

Then the monotony is established as follows:

Monotony: P1 ensures that $\succsim_P$ is transitive, so monotony can be soundly defined. Consider $A \subseteq N$ and $x, x' \in \mathcal{A}$ such that $\forall i \in A, x'(i) \succ_P x(i)$. By P3, we get $\forall i \in A, \forall z, x'\{i\}z \succsim x\{i\}z$ Since weak unanimity has been proved, it holds that $x'Az \succsim xAz, \forall z$. So, by P2, $x'Ax \succsim x$. When $x \succsim y$, we get $x'Ax \succsim y$ by transitivity which establishes LM. Similarly, for RM:, consider $y, y' \in X^S$ such that $\forall i \in A, y(i) \succ_P y'(i)$. Since weak unanimity holds, we get $yAz \succsim y'Az, \forall z$. So, by P2, $y \succsim y'Ay$. When $x \succsim y$, we get $x \succsim y'Ay$ by transitivity.

Proof of Theorem 2. The definition of $\succsim_P$ from $\succsim$ (equation 4) is self sufficient. When P3 and OI hold, $\succsim_L$ can also be soundly defined from $\succsim$ using definition 7.

Suppose first that P5 holds and consider two consequences a and b such that $a \succ_P b$. To get the main result, it is sufficient to show that, whatever i, $x(i) \succsim_P y(i) \Leftrightarrow (aC_{\succsim}(x,y)b)(i) \succsim_P (aC_{\succsim}(y,x)b)(i)$. Indeed, in that case, $(x,y) \equiv (aC_{\succsim}(x,y)b, aC_{\succsim}(y,x)b)$ (recall that $\succsim_P$ is complete) and OI would enable the deduction of $x \succsim y \Leftrightarrow aC_{\succsim}(x,y)b \succsim aC_{\succsim}(y,x)b$. By definition of $\succsim_L$, $C_{\succsim}(x,y) \succsim_L C_{\succsim}(y,x) \Leftrightarrow aC_{\succsim}(x,y)b \succsim aC_{\succsim}(y,x)b$. Hence, we would get $x \succsim y \Leftrightarrow C_{\succsim}(x,y) \succsim_L C_{\succsim}(y,x)$.

Let us show that, for all i, $(x(i) \succsim_P y(i) \Leftrightarrow (aC_{\succsim}(x,y)b)(i) \succsim_P (aC_{\succsim}(y,x)b)(i))$:
- Any i such that $x(i) \sim_P y(i)$ belongs to $C_{\succsim}(x,y)$ and to $C_{\succsim}(y,x)$. Hence $(aC_{\succsim}(x,y)b)(i) = (aC_{\succsim}(y,x)b)(i) = a$. So, by reflexivity of $\succsim$ and thus of $\succsim_P$ we have: $(aC_{\succsim}(x,y)b)(i) \sim_P (aC_{\succsim}(y,x)b)(i)$.
- Any i such that $x(i) \succ_P y(i)$ belongs to $C_{\succsim}(x,y)$ and does not belong to $C_{\succsim}(y,x)$. Hence $(aC_{\succsim}(x,y)b)(i) = a$ and $(aC_{\succsim}(y,x)b)(i) = b$. Since $a \succ_P b$, $(aC_{\succsim}(x,y)b)(i) \succ_P (aC_{\succsim}(y,x)b)(i)$.
- Similarly, for i such that $y(i) \succ_P x(i)$, we get $(aC_{\succsim}(y,x)b)(i) \succ_P (aC_{\succsim}(x,y)b)(i)$.
Since $\succsim_P$ is complete, this proves that $x(i) \succsim_P y(i) \Leftrightarrow C_{\succsim}(x,y)b(i) \succsim_P aC_{\succsim}(y,x)b(i)$.

On the other hand, suppose that P5 does not hold. Since $\succsim_P$ is complete, this means that: $\forall x,y \in \mathcal{A}, \forall i \in N$, $x(i) \sim_P y(i)$. Thus $\forall x,y, C_{\succsim}(x,y) = C_{\succsim}(y,x) = S$. Thus $\forall x,y, C_{\succsim}(x,y) \succsim_L C_{\succsim}(y,x)$ (reflexivity of $\succsim$ and thus of $\succsim_L$). Moreover, by OI and reflexivity of $\succsim$ we get $x \succsim y, \forall x,y$. From $[\forall x,y, C_{\succsim}(x,y) \succsim_L C_{\succsim}(y,x)$ and $\forall x,y, x \succsim y]$, we can derive $[\forall x,y, C_{\succsim}(x,y) \succsim_L C_{\succsim}(y,x) \Leftrightarrow x \succsim y]$.

Proof of Proposition 4. Under P5, $\exists a,b \in X$ such that $a \succ_P b$. So, $A \succsim_L B \Leftrightarrow aAb \succsim_L aBb$. Moreover, OI and $\succsim$ complete on constant acts imply P2 (Proposition 1).
Preadditivity of $\succsim_L$. Consider A,B,C such that $A \cap (B \cup C) = \emptyset$. $B \succsim_L C \Leftrightarrow aBb \succsim aCb$. Since $A \cap (B \cup C) = \emptyset$ and P2 holds, we have $aBb \succsim aCb \Leftrightarrow aBaAb \succsim aCaAb$, which means $B \succsim_L C \Leftrightarrow A \cup B \succsim_L A \cup C$.

Monotony of $\succsim_L$. Suppose that $A \succsim_L B$, i.e. $aAb \succsim aBb$. By LM, we get $aAa(\bar{A} \cap C)b \succsim aBb$ i.e. $A \cup C \succsim_L B$. Suppose that $A \succsim_L B \cup C$, i.e. $aAb \succsim a(B \cup C)b$. By RM, we get $aAb \succsim aBb$, i.e. $A \succsim_L B$.

Acceptance P5 requires that X contains at least three elements a,b,c such that $a \succ b$, $a \succ c$, $b \succ c$. So, by definition 5, $A \succsim_L B \Leftrightarrow (aAb) \succsim (aBb)$. Let $E,F,G \subseteq N$ be such that $E \cap F = E \cap G = F \cap G = \emptyset$, $E \cup G \succ_L F$, and $E \cup F \succ_L G$ and consider the six following acts: $ef = a(E \cup F)b$, $eg = a(E \cup G)b$, $fg = a(F \cup G)b$, $e = aEb$, $f = aFb$, $g = aGb$. Thanks to Theorem 2, $eg \succ f \Leftrightarrow E \cup G \succ_L F$ and $ef \succ g \Leftrightarrow E \cup F \succ_L G$, and therefore $eg \succ f$ and $ef \succ g$. Since E, F and G are disjoint, one can also consider the acts: $x = aEbFcGb$, $y = bEcFaGb$, $z = cEaFbGb$. It is easy to check that $(x,y) \equiv (ef,g)$ and $[(y,z) \equiv (eg,f)]$. Hence by OI we get $x \succ y$ and $y \succ z$. By transitivity of $\succ$, this implies that $x \succ z$. Since $(x,z) \equiv (a,bc)$, OI implies that $a \succ bc$ i.e. $aEb \succ a(F \cup G)b$. By Theorem 2, this is equivalent to $E \succ_L F \cup G$.

Proof of Proposition 5. Let $x,y \in X$ such that, for all non-null $\{j\} \subseteq N$, $x_j \succ_j y_j$. It is easy to check that $(x,y) \equiv (x^*,x_*)$ and $(xAy, xBy) \equiv (x^*Ax_*, x^*Bx_*)$. So, by OI : $(xAy) \succsim (xBy) \Leftrightarrow (x^*Ax_*) \succsim (x^*Bx_*)$. So, $\forall x,y \in X$ such that, for all non-null $\{j\} \subseteq N$, $x_j \succ_j y_j$: $(xAy) \succsim (xBy) \Leftrightarrow (x^*Ax_*) \succsim (x^*Bx_*)$. By definition 5, this proves $A \succsim_L B \Leftrightarrow (x^*Ax_*) \succsim (x^*Bx_*)$.

Proof of Theorem 3.
Completeness of $\succsim_j$. The proof is direct from MC. Indeed, MC means that for all $x,y \in \mathcal{A}$, and whatever $j \in N$ it holds that: $\forall z \in \mathcal{A}, (x\{j\}z) \succsim (y\{j\}z)$ or $\forall z \in \mathcal{A}, (y\{j\}z) \succsim (x\{j\}z)$. By definition (4), this means that $x_j \succsim_j y_j$ or $y_j \succsim_j x_j$. Since this holds for any pair (x,y), this proves that the $\succsim_j$, $j = 1,\ldots,n$ are complete.

Preadditivity of $\succsim_L$. Similar to the one of DMU, with x^* (resp. x_*) playing the role of the constant act a (resp. the constant act b). Indeed, P2 still holds in MCDM without any modification of its definition.

Main result. Let us now consider a pair $(v,w) \in \mathcal{A}^2$ such that $v \succsim w$. Let us denote $A = C_{\succsim}(v,w)$ and $B = C_{\succsim}(w,v)$. Consider the relation $\succsim_L$ between coalitions defined by equation (5). Since OI holds we know by Proposition 5 that $A \succsim_L B$ if and only if $(x^*Ax_*) \succsim (x^*Bx_*)$, Now, for any $j \in N$, $\succsim_j$ is complete. So, for criteria j, only 3 cases are possible:
- $v_j \succ_j w_j$: $(x^*Ax_*)_j = (x^*)_j$ and $(x^*Bx_*)_j = (x_*)_j$, thus $(x^*Ax_*)_j \succ_j (x^*Bx_*)_j$
- $w_j \succ_j v_j$: $(x^*Ax_*)_j = (x_*)_j$ and $(x^*Bx_*)_j = x^*_j$, thus $(x^*Bx_*)_j \succ_j (x^*Ax_*)_j$
- $v_j \sim_j w_j$: $(x^*Ax_*)_j = x^*_j$ and $(x^*Bx_*)_j = x^*_j$, thus $(x^*Ax_*)_j \sim_j (x^*Bx_*)_j$
So, $C_{\succ}(v,w) = C_{\succ}(x^*Ax_*, x^*Bx_*)$, $C_{\succ}(w,v) = C_{\succ}(x^*Bx_*, x^*Ax_*)$, $C_{\sim}(v,w) = C_{\sim}(x^*Ax_*, x^*Bx_*)$. Thus $C_{\succsim}(v,w) = C_{\succsim}(x^*Ax_*, x^*Bx_*)$ and $C_{\succsim}(w,v) = C_{\succsim}(x^*Bx_*, x^*Ax_*)$. So $(v,w) \equiv (x^*Ax_*, x^*Bx_*)$. By OI, $v \succsim w \Leftrightarrow (x^*Ax_*) \succsim (x^*Bx_*)$. Moreover, we know that $(x^*Ax_*) \succsim (x^*Bx_*) \Leftrightarrow A \succsim_L B$. Thus: $v \succsim w \Leftrightarrow C_{\succsim}(v,w) \succsim_L C_{\succsim}(w,v)$.

Proof of Proposition 6. The proof is similar to the one of DMU, with $x \succ y \succ z$ playing the roles of the constant act a, b and c. They indeed exist, due to DI.

Proof of Theorem 4. This theorem is an obvious consequence of Theorem 5 - just by letting $N' = N$. A direct proof can be obtained by transposing a result of Weymark [35] in Social Choice Theory to the case of a cartesian product.

Proof of Theorem 5. By Theorem 3, we know that $x \succsim y \Leftrightarrow C_{\succsim}(x,y) \succsim_L C_{\succsim}(y,x)$. We also proved (Proposition 6) that $\succsim_L$ is an acceptance relation. Its monotonicity can be easily derived from MO (with a proof similar to the proof of proposition 4).

Lemma 5.1 : If $\succsim$ satisfies A1, DI, MO and OI then:
$\forall O \subseteq N' \subseteq N$, O is decisive in $N' \Leftrightarrow O \succ_L N' \setminus O$

Proof: Suppose that O is decisive in N' and consider the acts $x^*Ox_*(N' \setminus O)z$ and $x^*(N' \setminus O)x_*Oz$. Since O is decisive in N', we get $x^*Ox_*(N' \setminus O)z \succ x^*(N' \setminus O)x_*Oz, \forall z$. So, it holds that $x^*Ox_*(N'\setminus O)x_* \succ x^*(N'\setminus O)x_*Ox_*$, i.e. $O \succsim_L N' \setminus O$.

Reciprocally, suppose that $O \subseteq N'$, $O \succ_L N' \setminus O$ and consider x, y such as $\forall i \in O, x(i) \succ_P y(i)$. Consider the acts $xN'z$ and $yN'z$. It holds that $O \subseteq C_{\succ}(xN'z, yN'z)$. Since $\succsim_P$ is complete, we also have: $C_{\succ}(yN'z, xN'z) \subseteq N \setminus O$ and thus $C_{\succ}(yN'z, xN'z) \subseteq N' \setminus O$. By monotony of $\succsim_L$, $O \succ_L N' \setminus O$ gives $C_{\succ}(xN'z, yN'z) \succ_L C_{\succ}(yN'z, xN'z)$. By additivity of $\succ_L$, this is equivalent to $C_{\succsim}(xN'z, yN'z) \succ_L C_{\succsim}(yN'z, xN'z)$, i.e. $xN'z \succ yN'z$. Since we did not make any assumption on z, this means $(x \succ y)_{N'}$.

Lemma 5.2 : If $\succsim$ satisfies A1, DI, MO and OI, then there exists at most one coalition predominant in N'

Proof: Suppose N' contains at least two distinct predominant coalitions O_1 and O_2.
On the one hand we get $x^*O_1x_*(N' \setminus O_1)z \succ x_*O_1x^*(N' \setminus O_1)z, \forall z$, since O_1 is decisive in N'.
On the other hand, since O_2 is predominant in N' and distinct from O_1, there exists at least one $i \in O_2 \setminus O_1$ that is a vetoer. Thus, there is at least one $i \in N' \setminus O_1$ that is a vetoer. This fact contradicts $x^*O_1x_*(N' \setminus O_1)z \succ x_*O_1x^*(N' \setminus O_1)z$.

Proof of the main result: Let us now remark that if $A \subseteq N'$ is decisive in N' and $B \subseteq N'$ is decisive in N', then $A \cap B$ is non empty and decisive in N'. Indeed, thanks to Lemma 5.1, this implies that $A \succ_L \bar{A} \cap N'$ and $B \succ_L \bar{B} \cap N'$. Thus $(A \cap B) \cup (A \cap \bar{B}) \succ_L (\bar{A} \cap \bar{B} \cap N') \cup (\bar{A} \cap B)$ and $(A \cap B) \cup (\bar{A} \cap B) \succ_L (\bar{A} \cap \bar{B} \cap N') \cup (A \cap \bar{B})$. By monotony of $\succsim_L$, we get $(A \cap B) \cup (A \cap \bar{B}) \succ_L (\bar{A} \cap \bar{B} \cap N') \cup (\bar{A} \cap B)$ and $(A \cap B) \cup (\bar{A} \cap B) \cup (\bar{A} \cap \bar{B} \cap N') \succ_L (A \cap \bar{B})$. By acceptance, we get: $(A \cap B) \succ_L (\bar{A} \cap \bar{B} \cap N') \cup (\bar{A} \cap B) \cup (A \cap \bar{B})$, i.e. $(A \cap B) \succ_L N' \setminus (A \cap B)$. So, $(A \cap B)$ is decisive in N'. It cannot be empty, otherwise we would get $\emptyset \succsim_L N'$ which contradicts the assumption that N' is not null.

Since N' is not null, $N' \succ_L \emptyset$ (applying Prop. 4). From Lemma 5.1, thus know that there exist at least one decisive coalition in N': N' itself. So, $O = \bigcap_{A, A \subseteq N', A \succ_L N' \setminus A}$, the intersection of all decisive sets in N' is a non empty subset of N' which is decisive in N'. Let us show that any $i \in O$ is a vetoer. Consider x, y such that $x(i) \succ_P y(i)$. i belongs to $C_{\succ}(xN'z, yN'z)$. Suppose that $yN'z \succ xN'z$, which is equivalent to $C_{\succ}(yN'z, xN'z) \succ_L C_{\succ}(xN'z, yN'z)$. Since $i \in C_{\succ}(xN'z, yN'z)$ and $C_{\succ}(yN'z, xN'z) \subseteq N' \setminus \{i\}$, the monotony of $\succsim_L$ implies $N' \setminus \{i\} \succ_L \{i\}$. So, i cannot belong to $O = \bigcap_{A, A \succ_L N' \setminus A}$. This yields a contradiction.So, i must be a vetoer.
$O = \bigcap_{A, A \succ_L N' \setminus A}$ is thus a set predominant in N'. By Lemma 5.2, we know that it is unique.

Proof of the additional property : By definition of O, if $A \succ_L N' \setminus A$, then $O \subseteq A$. Conversely, if $O \subseteq A$, $O \succ_L N' \setminus O$ implies by monotony of $\succsim_L$ that $A \succ_L N' \setminus A$.

Proof of Corollary 2. The existence of the top predominant set O_1 is due to Theorem 5 (just by letting $N' = N$). Now, suppose that the property holds until level j. So, there exists a subset O_j predominant in $N_j = N \setminus (O_1 \cup \cdots \cup O_{j-1})$. Then, two cases must be distinguished:
i) If $N_{j+1} = N_j \setminus O_j$ is null, then the hierarchy is: $O_1, \ldots, O_j$ and $O_{j+1} = N_{j+1}$.
ii) If N_{j+1} is not null, then we know, thanks to Theorem 5, that there exists a unique O_{j+1} predominant in N_{j+1}. This process can be iterated until the set of remaining states is a null set.

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