Can qualitative Utility Criteria obey the Sure Thing Principle ?

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Abstract

In the framework of Artificial Intelligence it has been pointed out that information about preferences and uncertainty in decision problems cannot always be quantified but only qualitative evaluations can sometimes be attained. That is why two qualitative decision criteria based on possibilistic representations of uncertainty and preferences have been proposed and axiomatically justified. They are particular cases of a wider family of decision criteria called *monotonic utilities* that is based on a Sugeno integral.

These criteria suffer from a double drawback: first, monotonic utilities in general and possibilistic utilities in particular, may suffer from a lack of decisiveness power. Especially, when two actions give an identical and extreme (either good or bad) consequence in some given, plausible state of the world, they may not be distinguished by possibilistic decision criteria, although they may give significantly different consequences in the other states of the world. Second, the possibilistic utility criteria do not fully satisfy the Sure-Thing Principle (STP), which is widely accepted as a fundamental property of decision criteria.

In this paper, we try to cope with these two problems, proposing refinements of the possibilistic decision criteria that obey the Sure Thing Principle. These two procedures are derived from the comparison of the monotonic utilities of acts, every state in which both acts give identical results, being neglected. We then investigate the more general family of criteria that is obtained when one apply this discrimination procedure to monotonic utilities.

1 Introduction

In the framework of Artificial Intelligence, it has been pointed out that information about preference and uncertainty in decision problems cannot always be quantified in a simple way, but only qualitative evaluations can sometimes be attained. As a consequence, the topic of qualitative decision theory is a natural one to consider [3, 2, 1, 12]: can we make decision on the basis of qualitative information? In this kind of research the set of states of the world and the set of consequences of actions are often supposed to be finite, contrary to classical frameworks where one of them is infinite. Moreover, Giving up the quantification of utility and uncertainty also means that the notion of expectation based on averaging is given up as well. It contrasts with non-expected utilities based on Choquet integrals where the idea of average value of a decision is preserved although it becomes imprecise [10, 14].

A trade-off between purely symbolic and purely numerical approaches to ranking decisions consists in using a single qualitative scale for assessing beliefs and utilities. In [7] two qualitative criteria, an optimistic and a pessimistic one, whose definitions only require a linearly ordered scale for utility and uncertainty are proposed as well as axiomatic justifications for them in the style of von Neumann and Morgenstern [16] and more recently, [8, 9], in the style of Savage [13]. In these qualitative criteria, uncertainty and preferences are captured by possibility distributions [18, 6], and more precisely in the framework of qualitative possibility theory [4]. The pessimistic criterion can again be viewed as a refinement of the Wald criterion, but these qualitative utilities reject the notion of expectation put forward by the classical theory, and also sanctioned by Choquet integrals. This framework is tailored for one-shot (non repeated) decisions, where the decision is evaluated by means of a particular state prevision that belongs to the ones considered as plausible for the decision-maker. Later, a wider family of decision criteria called *monotonic utilities* and that embeds both possibilistic decision criteria, was proposed [9]. It is based, mathematically speaking, on a Sugeno integral which is a qualitative counterpart of the Choquet integral, generalizing expected utility.

Unfortunately, monotonic utilities in general and possibilistic utilities in particular, may suffer from a lack of decisiveness power in some circumstances. Especially, when two actions give an identical and extreme (either good or bad) consequence in some given, plausible state of the world, they may not be distinguished by possibilistic decision criteria, although they may give significantly different consequences in the other states of the world. Also, the possibilistic utility criteria do not fully satisfy the Sure-Thing Principle (STP), which is widely accepted as a fundamental property of decision criteria, and that forbids this kind of lack of decisiveness power.

So, the point of the article is to give some tools in order to cope with this lack of decisiveness of possibilistic decision criteria. In particular, it leads us to define a family of decision procedures that we call *discriminant monotonic decision procedures* that all satisfy the STP, and therefore, gain in decisiveness power. These procedures are derived from the comparison of the monotonic utilities of acts, every state in which both acts give identical results being neglected. Among this family we will try to exhibit the decision procedures that possess the following, desirable properties:

- First, they shall refine the monotonic utility criterion from which they are derived,
- second, they shall at least preserve the acyclicity of the strict preference over acts.

In section 2, we will first give some background on monotonic and possibilistic utilities, and point out the "drowning effect" that may generate a lack of decisiveness power. Finally, in this section we will exhibit a kind of "impossibility theorem", that is that the only possibilistic decision criteria that obey the STP are degenerate. Then, in section 3 we will show two simple procedures, derived from the possibilistic decision criteria, that obey the STP and thus allow to cope with the drowning effect of possibilistic criteria. These two procedures are members of a general family, of which we will study the properties in section 4 and that can be derived from the family of monotonic utilities. In particular, we will study the desirable decisiveness and transitivity properties that will allow us to highlight an interesting subclass of decision procedures.

In section 5, we will shortly get an insight into another interesting subclass of decision procedures, in which uncertainty is modelled by qualitative beliefs and plausibilities, instead of necessities and possibilities.

2 Possibilistic decision-making

2.1 Possibilistic utility criteria

Savage has proposed a framework for axiomatizing decision rules under uncertainty where both the uncertainty function and the utility function are derived from first principles bearing on acts. In Savage's approach a preference relation $\preceq$ between acts (or decisions) is assumed to be given by a decision-maker.

Let us denote $\mathcal{F} = X^S$ the set of potential acts. The set of actually feasible acts is generally only a subset of $\mathcal{F}$. The first assumption made by Savage is that the preference relation on $\mathcal{F}$ is transitive and complete. Two particular families of acts are crucial to recover the preference information on consequences and the uncertainty information on the state space S: constant acts and binary acts respectively. A *constant act*, denoted $\mathbf{x}$ for $x \in X$ is such that $\forall s \in S, \mathbf{x}(s) = x$. Since $\preceq$ is a complete preorder on $\mathcal{F}$, the set of acts, it is also a complete preorder on the set of constant acts (which can be identified with X). Therefore, we can define a complete preorder $\leq_P$ on X from the restriction of $\preceq$ to constant acts.

The ranking of acts also induces a ranking of events, i.e., subsets of the state space: this is based on the use of binary acts. A *binary act* is an act f such that there is a set $A \subseteq S$ and two consequences $y <_P x \in X$ where $f(s) = x$ if $s \in A$ and $f(s) = y$ if $s \in \overline{A}$, where $\overline{A}$ is the complement of A. Such a binary act is denoted xAy. A partial ordering $\trianglerighteq$ of events can be defined by restricting the complete preordering on acts to binary acts.

The notion of binary act is a particular case of a compound act:

Definition 1 *Compound act: $\forall A \subseteq S$, fAg is the act defined by: $fAg(s) = f(s)$ for all $s \in A$, and $fAg(s) = g(s)$ for all $s \in \overline{A}$.*

Consider again the set $\mathcal{F}$ of acts f, mappings from S to X. Clearly if we take the Savagean framework for granted, it is possible to evaluate uncertainty and preference by means of a totally ordered scale $(L, \leq)$ (finite, as we will assume in this paper that S and X are finite). The mapping from the set of consequences to L is a utility function $\mu : X \to L$. It is supposed that the top and bottom elements of L, denoted 1 and 0 respectively, are in $\mu(X) = \{\mu(x), x \in X\}$. If not, just add an ideal consequence x^* and a totally bad consequence x_* to X. Uncertainty is supposed to be captured by means of a set function $\sigma : S \to L$ which is a monotonic measure (called *capacity*), that is such that: $\sigma(\emptyset) = 0$, $\sigma(S) = 1$, $A \subseteq B \Rightarrow \sigma(A) \leq \sigma(B)$. This kind of set-function is very general and represents the minimal requirement for the representation of partial belief. Especially the last condition is called *monotonicity*, and is verified by probability measures and most other well-known representations of partial belief (including belief and plausibility functions, necessity and possibility measures...). Then the utility of an act f can be defined as a Sugeno integral [15], a qualitative counterpart of expected sum, where the sum is replaced by a *sup* (a *max* in the finite case) and the product by a *inf* (a *min* in the finite case).

Definition 2 *Monotonic qualitative utility of an act* $U_\sigma(f) = \max_{s \in S} \min(\mu(f(s)), \sigma(F_s))$ *where* $F_s = \{s' \in S,\ \mu(f(s')) \geq_P \mu(f(s))\}$ *and* σ *is a monotonic measure.*

We can notice that $U_\sigma(f) = \max_{x \in X} \min(\mu(x), \sigma(F_x)) = \max_{\lambda \in L} \min(\lambda, \sigma(F_\lambda))$ where $F_x = \{s \in S,\ f(s) \geq_P x\}$ and $F_\lambda = \{s \in S,\ \mu(f(s)) \geq_L \lambda\}$. This Sugeno integral thus defines a complete preorder $\succeq_\sigma$ representing the preferences of the Decision Maker over acts:

Definition 3 *Preorder on actions derived from the monotonic qualitative utility:*
$f \succeq_\sigma g \Leftrightarrow U_\sigma(f) \geq U_\sigma(g)$.

Two qualitative possibilistic criteria for decision making that are special cases of Monotonic utility were proposed by [7]. These two criteria are respectively obtained when σ is forced into a possibility or a necessity measure.

Definition 4 *Possibilistic utilities:*
- Optimistic possibilistic utility:

$$U_\Pi(f) = \max_{s \in S} \min(\pi(s), \mu(f(s))),$$

- Pessimistic possibilistic utility:

$$U_N(f) = \min_{s \in S} \max(n(\pi(s)), \mu(f(s))).$$

Where π *is a possibility distribution over* S *and* μ *a qualitative utility function over* X*, both taking their values in a finite totally-ordered scale* L*.*

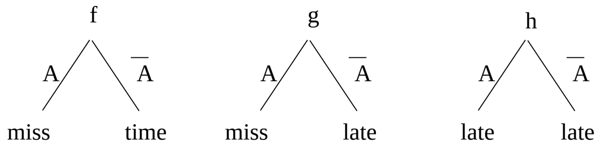
The second criteria can be seen as a refinement of the Wald criterion, which proposes that the utility of an act be that of its worst possible consequence. Thus, U_N is "pessimistic" or "cautious", even if the pessimism is moderated by taking relative possibilities of states into account. $U_N(f)$ is high only if f gives good consequences in every "rather plausible" state. On the other side, U_Π is a mild version of the *max* criterion which is "optimistic", or "adventurous" since an act is supposed to be "good", as soon as there exists a plausible state in which it gives a good result. Both criteria where axiomatized in terms of preference over possibilistic lotteries by [7], and by [8, 9] in terms of preference over acts.

Notice that two complete preorders on actions, $\succeq \Pi$ and $\succeq_N$ can be defined from U_Π and U_N, in the same way as for U_σ in general.

2.2 The "drowning effect"

Both possibilistic criteria suffer from an important drawback: They have to face what we call a "drowning effect". Since actions are evaluated with respect to the satisfaction degree and the possibility degree of only one possible consequence, it is possible that two actions leading to very different consequences, but to a common completely plausible and extreme (either bad or good) one, maybe found equivalent, just because the common consequence blurs all the differences.

Example 1 *Someone has to attend an important appointment at the other end of the city. The problem is to get there on time, the uncertainty lies in the uncertain event A:* "There are traffic jams". *Three routes are allowed. Route f: Driving through the town center, if there are no traffic jams Someone will be on time, if there are traffic jams the appointment will be missed. Route g: Bypassing the city by the east, in this case Someone will at best be late and will miss the appointment if there are traffic jams. Route h: Bypassing the city by the west, a long but safer way by which Someone will always be late but never miss the appointment.*



Trying to maximize the pessimistic utility, Someone will be indifferent between Route f and Route g (and strictly prefer Route h to both) despite the fact that Route f strictly dominates Route g in case there is no traffic, because the possibility of missing the appointment in case of traffic blurs all differences. A similar drowning effect occurs for an "optimistic" agent who will prefer Route f to both Routes g and h.

This drowning effect is observed when two actions give the same extremely good or bad consequence in one particular state, as for f and g, despite the fact that one action would be preferred to the other if this very state did not occur. It is not observed within expected utility theory since the following *Sure-Thing Principle* (STP) [13] insures that identical consequences do not influence the relative preference between two events.

$$\text{STP: } \forall f, g, h, h', fAh \succeq gAh \Leftrightarrow fAh' \succeq gAh'$$

Our motivation will be, in the remainder of the paper, to see whether it is possible in the possibilistic framework, to benefit from the STP, in order to escape from this "drowning effect". As a first step, we will show an "impossibility result": The only possibilistic criteria that obey the STP are "degenerate", in so far as they can not handle uncertainty. Nevertheless, we will be able to exhibit two interesting families of decision procedures that both obey the STP and refine respectively the procedures based on the possibilistic criteria.

2.3 Qualitative possibilistic criteria are incompatible with the sure thing principle

Let us show that only trivial cases of possibilistic decision criteria satisfy the Sure-thing principle:

Proposition 1 *Let $\succeq_\Pi$ (resp. $\succeq_N$) be modeled by a qualitative possibilistic optimistic (resp. pessimistic utility). $\succeq_\Pi$ (resp. $\succeq_N$) satisfies the STP iff $\exists! s^*$: $\pi(s*) = 1$ and $\forall s \neq s^*, \pi(s) = 0$.*

Proof: (for $\succeq_\Pi$)
The *if* part is trivial to show.
Only if part: Suppose that there exist s_1, s_2, $s_1 \neq s_2$, $\pi(s_1) \geq \pi(s_2) > 0$. Then, we have $1\{s_2\}0 \succ_\Pi \mathbf{0}$ and $1\{s_1, s_2\}0 \preceq_\Pi 1\{s_1\}0$, which is a violation of the STP. So, if $\succeq_\Pi$ satisfies the STP then there exists at most one s^* such that $\pi(s^*) > 0$. But, since π is normalized, such an s^* exists and $\pi(s^*) = 1$. □

Hence adding the STP to the current axiomatization of possibilistic utilities leads to focus the comparison of actions on one single state, independent of the actions, that is taken as the "real" state of the world : all the other states are null, i.e. impossible. In other terms, the only case where the pessimistic (resp. optimistic) utility criterion can obey the STP is the case where there is no uncertainty at all !

This result can be compared to the one obtained by Marichal [11] in the context of multicriteria decision making : it is shown there that the preferential independence condition (a multicriteria counterpart of the STP) is compatible with an aggregation of the preference based on a Sugeno integral only if one criterion acts like a strong dictator. In our context, the notion of strong dictatorship corresponds to a notion of full certainty.

3 A general family of decision procedures derived from the monotonic utilities and satisfying the Sure-Thing Principle

3.1 Refining the possibilistic utility criteria: Discriminant possibilistic utility criteria

In order to escape from the drowning effect, it is tempting to modify U_Π and U_N in such a way that they obey the STP. Let us recall that $U_\Pi(f) = \max_{s \in S} \min\{\pi(s), \mu(f(s)\}$. A simple way to restore the STP is to limit the comparison of two acts to the states in which they give different consequences. This leads to defining the following *Discri-opt* order: $\succeq_{\Pi\Delta}$

Definition 5 *Discri-opt order:*

$$f \succeq_{\Pi\Delta} g \Leftrightarrow \max_{s \in \Delta(f,g)} \min(\pi(s), \mu(f(s)))$$

$$\geq \max_{s \in \Delta(f,g)} \min(\pi(s), \mu(g(s)))$$

where $\Delta(f, g) = \{s \in S, f(s) \neq g(s)\}$. Furthermore, $\Delta(f, g) = \emptyset \Rightarrow f \sim_{\Pi\Delta} g$.

In the same way, one can define the *Discri-pes* order: $\succeq_{N\Delta}$

Definition 6 *Discri-pes order:*

$$f \succeq_{N\Delta} \Leftrightarrow \min_{s \in \Delta(f,g)} \max(n(\pi(s)), \mu(f(s)))$$

$$\geq \min_{s \in \Delta(f,g)} \max(n(\pi(s)), \mu(g(s)))$$

Furthermore, $\Delta(f, g) = \emptyset \Rightarrow f \sim_{N\Delta} g$.

One can notice that the drowning effect disappears in Example 1 when $\succ_{\Pi\Delta}$ or $\succ_{N\Delta}$ are used instead of $\succ_\Pi$ or $\succ_N$: $f \succ_{\Pi\Delta} h \succ_{\Pi\Delta} g$ and $h \succ_{N\Delta} f \succ_{N\Delta} g$. More generally, it can be proved that $\succ_{\Pi\Delta}$ refines $\succ_\Pi$ and $\succ_{N\Delta}$ refines $\succ_N$:

Proposition 2 *$f \succ_\Pi g \Rightarrow f \succ_{\Pi\Delta} g$ and $f \succ_N g \Rightarrow f \succ_{N\Delta} g$*

Proof: (for $\succ_{\Pi\Delta}$)
$f \succ_\Pi g \Rightarrow \exists s^* \in S, \min(\pi(s^*), \mu(f(s^*))) > \min(\pi(s), \mu(g(s))), \forall s \in S$. So, $\min(\pi(s^*), \mu(f(s^*))) > \min(\pi(s), \mu(g(s))), \forall s \in \Delta(f, g)$ since $\Delta(f, g) \subseteq S$. But $s^* \in \Delta(f, g)$, since, if it is not the case, we cannot have $\min(\pi(s^*), \mu(f(s^*))) > \min(\pi(s^*), \mu(g(s^*)))$. Finally, we have : $\exists s^* \in \Delta(f, g), \min(\pi(s^*), \mu(f(s^*))) > \min(\pi(s), \mu(g(s))), \forall s \in \Delta(f, g)$, that is: $f \succ_{\Pi\Delta} g$. □

Since $\Delta(fAh, gAh) \subseteq A$, it is also easy to prove that both $\succeq_{\Pi\Delta}$ and $\succeq_{N\Delta}$ satisfy the STP:

Proposition 3 *$\succeq_{\Pi\Delta}$ and $\succeq_{N\Delta}$ satisfy the STP.*

3.2 A general family of decision procedures derived from monotonic utilities

Actually, $\succeq_{N\Delta}$ and $\succeq_{\Pi\Delta}$ are particular cases of a more general family of decision procedures that can be derived in a simple way from the family of monotonic utilities.

In order to get this new family, we change the decision procedure based on the comparison of the monotonic utilities of two acts, by forcing the two acts to take a given fixed value on those states where they give identical consequences.

The obtained orderings on acts are indexed by this given value:

Definition 7 *Discri-monotonic decision procedures. Let σ be an ordinal capacity, and $\alpha \in L$ be a given constant value: $\succeq_{\sigma\Delta\alpha}$ is defined by:*

$$f \succeq_{\sigma\Delta\alpha} g \Leftrightarrow U_\sigma(f\Delta\alpha) \geq U_\sigma(g\Delta\alpha)$$

where $\Delta = \Delta(f, g)$.

In particular, we can notice that $\succeq_{\Pi\Delta} \equiv \succeq_{\Pi\Delta 0}$ and $\succeq_{N\Delta} \equiv \succeq_{N\Delta 1}$.

In the following section we will have a look on the properties of the Discri-monotonic decision procedures, as well as on the induced relations on events.

3.3 Projection of the discri-monotonic decision procedures on events

First of all, we shall notice that every $\succeq_{\sigma\Delta\alpha}$ relations satisfy the STP:

Proposition 4 *$\forall \sigma, \alpha, \succeq_{\sigma\Delta\alpha}$ satisfy the STP.*

Proof:
Just notice that for all A, f, g, h, h', $fAh \succeq_{\sigma\Delta\alpha} gAh \Leftrightarrow fA\alpha \succeq_{\sigma\Delta\alpha} gA\alpha \Leftrightarrow fAh' \succeq_{\sigma\Delta\alpha} gAh'$ □

Since discri-monotonic decision procedures obey the sure-thing principle, it is possible, as did Savage in the expected-utility framework, to derive from the preferences over acts an uncertainty relation over events (i.e. subsets of the state space), by the means of binary acts, in the following way:

Definition 8 *Uncertainty relation:*
$\forall A, B \subseteq S, A \trianglerighteq_{\sigma\Delta\alpha} B \Leftrightarrow \forall x > y, xAy \succeq_{\sigma\Delta\alpha} xBy$.

Since the STP holds, it is easy to check that $\trianglerighteq_{\sigma\Delta\alpha}$ is well-defined.

The *discrimax likelihood* relations of [5] are special cases of these projections: $\trianglerighteq_{\Pi\Delta 0}$ and $\trianglerighteq_{N\Delta 1}$. Notice, that $A \trianglerighteq_{\Pi\Delta 0} B \Leftrightarrow \Pi(A \cap \bar{B}) \geq \Pi(\bar{A} \cap B) \Leftrightarrow N(A \cup \bar{B}) \geq N(\bar{A} \cup B) \Leftrightarrow A \trianglerighteq_{N\Delta 1} B$ (see [5] for the properties of the discrimax likelihood relations).

More generally, $A \trianglerighteq_{\sigma\Delta 0} B \Leftrightarrow \sigma(A \cap \bar{B}) \geq \sigma(\bar{A} \cap B)$ and $A \trianglerighteq_{\sigma\Delta 1} B \Leftrightarrow \sigma(A \cup \bar{B}) \geq \sigma(\bar{A} \cup B)$. The following general property holds:

Proposition 5 $\trianglerighteq_{^T\sigma\Delta\alpha} \equiv \trianglerighteq_{\sigma\Delta n(\alpha)}$ *where $^T\sigma(A) = n(\sigma(\bar{A}))$.*

Proof: $A \trianglerighteq_{\sigma\Delta n(\alpha)} B \Leftrightarrow \max\{\min(n(\alpha), \sigma(A \cap \bar{B})), \sigma(A \cup \bar{B})\} \geq \max\{\min(n(\alpha), \sigma(\bar{A} \cap B)), \sigma(\bar{A} \cup B)\}$ $\Leftrightarrow n(\max\{\min(n(\alpha), \sigma(\bar{A} \cap B)), \sigma(\bar{A} \cup B)\}) \geq n(\max\{\min(n(\alpha), \sigma(A \cap \bar{B})), \sigma(A \cup \bar{B})\})$.
But, $n(\max\{\min(n(\alpha), \sigma(\bar{A} \cap B)), \sigma(\bar{A} \cup B)\}) = \min\{\max(\alpha, n(\sigma(\bar{A} \cap B))), n(\sigma(\bar{A} \cup B))\} = \max\{\min(\alpha, n(\sigma(\bar{A} \cup B)), \min(n(\sigma(\bar{A} \cap B)), n(\sigma(\bar{A} \cup B))\} = \max\{\min(\alpha, n(\sigma(\bar{A} \cup B)), n(\sigma(\bar{A} \cap B))\}$ (since $\bar{A} \cap B \subseteq \bar{A} \cup B$ and σ is monotonic) $= \max\{\min(\alpha, {}^T\sigma(A \cap \bar{B})), {}^T\sigma(A \cup \bar{B})\}$.
In the same way, $n(\max\{\min(n(\alpha), \sigma(A \cap \bar{B})), \sigma(A \cup \bar{B})\}) = \max\{\min(\alpha, {}^T\sigma(\bar{A} \cap B)), {}^T\sigma(\bar{A} \cup B)\}$, so that $A \trianglerighteq_{\sigma\Delta n(\alpha)} B \Leftrightarrow A \trianglerighteq_{^T\sigma\Delta\alpha} B$. □

4 Discri-monotonic decision procedures

4.1 General properties of the $\succeq_{\sigma\Delta\alpha}$ decision procedures

It is interesting to notice that $U_\sigma(f\Delta\alpha)$ may be rewritten as a function of $U_\sigma(f\Delta 0)$ and $U_\sigma(f\Delta 1)$ where 0 and 1 are two extreme consequences (of utility 0_L and 1_L respectively).

Proposition 6 *$\forall f, \alpha$ and $\forall \Delta \subseteq S$, $U_\sigma(f\Delta\alpha) = \max\{\min(\alpha, U_\sigma(f\Delta 1)), U_\sigma(f\Delta 0)\}$.*

Proof: $U_\sigma(f\Delta\alpha) = \max_{\lambda \in L} \min(\lambda, \sigma([f\Delta\alpha]_\lambda))$ where $[f\Delta\alpha]_\lambda = \{s \in S, (f\Delta\alpha)(s) \geq \lambda\}$, that is $[f\Delta\alpha]_\lambda = F_\lambda \cap \Delta$ if $\alpha < \lambda$ and $[f\Delta\alpha]_\lambda = F_\lambda \cup \Delta$ if $\alpha \geq \lambda$. So, $U_\sigma(f\Delta\alpha) = \max\{\max_{\lambda \leq \alpha} \min(\lambda, \sigma(F_\lambda \cup \Delta)), \max\{\max_{\lambda > \alpha} \min(\lambda, \sigma(F_\lambda \cap \Delta))\}$. But, $\forall \lambda \leq \alpha, \min(\lambda, \sigma(F_\lambda \cup \Delta)) \geq \min(\lambda, \sigma(F_\lambda \cap \Delta))$, since $F_\lambda \cap \Delta \subseteq F_\lambda \cup \Delta$. So, $U_\sigma(f\Delta\alpha) = \max\{\max_{\lambda \leq \alpha} \min(\lambda, \sigma(F_\lambda \cup \Delta)), U_\sigma(f\Delta 0)\}$. We also have $\max\{\max_{\lambda \leq \alpha} \min(\lambda, \sigma(F_\lambda \cup \Delta)) = U_\sigma(f\Delta 1)$ if $U_\sigma(f\Delta 1) \leq \alpha$ and α if not. Finally, $U_\sigma(f\Delta\alpha) = \max\{\min(\alpha, U_\sigma(f\Delta 1)), U_\sigma(f\Delta 0)\}$□

The following corollary can be deduced from the expression of $U_\sigma(f\Delta\alpha)$:

Corollary 1 *$\forall \alpha_1 \leq \alpha \leq \alpha_2 \in L, (f \succeq_{\sigma\Delta\alpha_1} g$ and $f \succeq_{\sigma\Delta\alpha_2} g) \Rightarrow f \succeq_{\sigma\Delta\alpha} g$.*

Unfortunately, as we are going to see, the decision procedures $\succ_{\sigma\Delta\alpha}$ are neither transitive (even not acyclic), nor do they refine the monotonic utility-based decision procedures they are derived from.

Proposition 7 *$\succ_{\sigma\Delta 0}$ and $\succ_{\sigma\Delta 1}$ are not transitive in general. That is, we can have $f \succ_{\sigma\Delta 0} g \succ_{\sigma\Delta 0} h \succ_{\sigma\Delta 0} f$ or $f \succ_{\sigma\Delta 1} g \succ_{\sigma\Delta 1} h \succ_{\sigma\Delta 1} f$.*

Proof: (for $\succ_{\sigma\Delta 0}$) Just take σ such that for $A, B, C \subseteq S$, $\sigma(A \cap \bar{B}) = \sigma(B \cap \bar{C}) = \sigma(\bar{A} \cap C) = \sigma(S) = 1$ and $\sigma(X) = 0$ for every other subsets, and $f = 1A0, g = 1B0$ and $h = 1C0$. □

As a result, $\succeq_{\sigma\Delta\alpha}$ is not transitive in general.

Proposition 8 *$\succ_{\sigma\Delta 0}$ and $\succ_{\sigma\Delta 1}$ do not refine $\succ_\sigma$ in general. In particular, we can have $g \succ_\sigma f$ and $f \succ_{\sigma\Delta 0} g$ or $f \succ_{\sigma\Delta 1} g$.*

Proof: (for $\succ_{\sigma\Delta 0}$) Just take σ such that $\sigma(A \cap \bar{B}) > \sigma(\bar{A} \cap B)$ and $\sigma(B) > \sigma(A)$. Such a σ may be found easily, since capacities are very general set-functions. Obviously, $f \succ_{\sigma\Delta 0} g$ and $g \succ_\sigma f$. □

We can notice that even the restrictions $\trianglerighteq_{\sigma\Delta\alpha}$ of $\succeq_{\sigma\Delta\alpha}$ on events do not possess the acyclicity and refinement properties:

Corollary 2 *$\triangleright_{\sigma\Delta\alpha}$ is not acyclic and does not refine $\triangleright_\sigma$ in general.*

Proof: (for $\triangleright_{\sigma\Delta 0}$)
Just notice that the two preceding proofs can be rewritten exactly in the same way, in terms of comparison of events. □

Recall that we are interested in decision procedures that:

- refine $\succeq_\Pi$ or $\succeq_N$,
- are at least *acyclic*, that is we cannot have $f \succ g \succ h \succ f$.

In the following we will focus our attention on the subclass of Discri-possibilistic decision procedures ($\sigma = \Pi$ or $\sigma = N$). We will show under which conditions they refine respectively the possibilistic optimistic and pessimistic decision procedures, and under which conditions they are acyclic. Especially, we will study the influence of the α parameter.

4.2 The Discri-possibilistic decision procedures

In this section we will study some properties of the $\succeq_{\Pi\Delta\alpha}$. Similar properties may be derived for $\succeq_{N\Delta\alpha}$, by duality.

4.2.1 When does $\succ_{\Pi\Delta\alpha}$ refine $\succ_\Pi$?

First of all, as a particular case of σ, notice that

$$U_\Pi(f\Delta\alpha) = \max\{\min(\alpha, U_\Pi(f\Delta 1)), U_\Pi(f\Delta 0)\}.$$

But, $U_\Pi(f\Delta 0) = \max_{s \in \Delta} \min(\pi(s), \mu(f(s)))$ and $U_\Pi(f\Delta 1) = \max\{\max_{s\in\Delta} \min(\pi(s), \mu(f(s))), \max_{s\in\bar{\Delta}} \pi(s)\} = \max\{U_\Pi(f\Delta 0), \Pi(\bar{\Delta})\}$, so that:

$$U_\Pi(f\Delta\alpha) = \max\{U_\Pi(f\Delta 0), \min(\alpha, \Pi(\bar{\Delta}))\}.$$

Then, the following proposition becomes easy to prove:

Proposition 9 *$\succ_{\Pi\Delta 0}$ refines $\succ_{\Pi\Delta 1}$, i.e. $f \succ_{\Pi\Delta 1} g \Rightarrow f \succ_{\Pi\Delta 0} g$.*

And more generally:

Proposition 10 *$\forall \alpha \leq \alpha', \succ_{\Pi\Delta\alpha}$ refines $\succ_{\Pi\Delta\alpha'}$, that is $f \succ_{\Pi\Delta\alpha'} g \Rightarrow f \succ_{\Pi\Delta\alpha} g$.*

Proof: $f \succ_{\Pi\Delta\alpha'} g \Leftrightarrow \max\{\min(\alpha', \Pi(\bar{\Delta})), U_\Pi(f\Delta 0)\} > \max\{\min(\alpha', \Pi(\bar{\Delta})), U_\Pi(g\Delta 0)\}$
$\Leftrightarrow U_\Pi(f\Delta 0) > \max\{\min(\alpha', \Pi(\bar{\Delta})), U_\Pi(g\Delta 0)\}$
$\Rightarrow U_\Pi(f\Delta 0) > \max\{\min(\alpha, \Pi(\bar{\Delta})), U_\Pi(g\Delta 0)\}$, since $\alpha \leq \alpha'$. So, $\max\{\min(\alpha, \Pi(\bar{\Delta})), U_\Pi(f\Delta 0)\} > \max\{\min(\alpha, \Pi(\bar{\Delta})), U_\Pi(g\Delta 0)\}$, that is $f \succ_{\Pi\Delta\alpha'} g$. □

As a corollary, we get:

Corollary 3 *$\succ_{\Pi\Delta 0}$ refines $\succ_\Pi$ refines $\succ_{\Pi\Delta 1}$.*

At this point, two questions maybe raised about the choice of an α, according to the problem of refinement:

- Does it exist an α, such that whatever the uncertainty model π, $\succ_{\Pi\Delta\alpha}$ refines $\succ_\Pi$?
- An uncertainty model π and a preference model μ being given, what is the exact value α^* such that $\forall\alpha \leq \alpha^*$, $\succ_{\Pi\Delta\alpha}$ refines $\succ_\Pi$ and $\forall\alpha > \alpha^*$, $\succ_\Pi$ refines $\succ_{\Pi\Delta\alpha}$?

The answer of the first question is in Corollary 3. Obviously it is "yes" since $\alpha = 0$ provides such a value. Still, we have no general answer to the second question.

4.2.2 When is $\succ_{\Pi\Delta\alpha}$ transitive?

We have seen before that in general $\succ_{\sigma\Delta\alpha}$ is not transitive, and even not acyclic. In this section, we are going to see that the choice $\sigma = \Pi$ improves things, that is : $\succ_{\Pi\Delta\alpha}$ is acyclic in general, and under some restrictive condition on α, $\succ_{\Pi\Delta\alpha}$ is even transitive.

Proposition 11 *$f \succ_{\Pi\Delta\alpha} g$ and $g \succ_{\Pi\Delta\alpha} h$ implies $f \succeq_{\Pi\Delta\alpha} h$.*

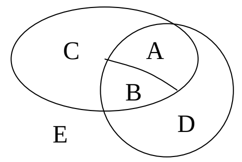
Proof: The notations are the same as for the preceding proof. $f \succ_{\Pi\Delta\alpha} g \Leftrightarrow \max\{\min(\alpha, \Pi(D \cup E)), F_{A\cup B\cup C}\} > \max\{\min(\alpha, \Pi(D \cup E)), G_{A\cup B\cup C}\} \Rightarrow F_{A\cup B\cup C}\} >$

$G_{A\cup B\cup C}\}$. In the same way, $g \succ_{\Pi\Delta_\alpha} h \Rightarrow G_{A\cup B\cup D}\} > H_{A\cup B\cup D}\}$. As before, from these two inequalities, we get $\max\{\max(F_A, G_{A\cup B}), F_{B\cup C\cup D}\} > \max\{\max(F_A, G_{A\cup B}), H_{B\cup C\cup D}\}$ and $F_{B\cup C\cup D} > H_{B\cup C\cup D}\}$, from which $\max\{\min(\alpha, \Pi(A\cup E)), F_{B\cup C\cup D}\} \geq \max\{\min(\alpha, \Pi(A\cup E)), H_{B\cup C\cup D}\}$, that is $f \succeq_{\Pi\Delta_\alpha} h$. □

Now, the question is: is there a condition under which $\succ_{\Pi\Delta_\alpha}$ is transitive? The answer is yes and we prove that $\succ_{\Pi\Delta 0}$ is transitive:

Proposition 12 $\succ_{\Pi\Delta 0}$ *is transitive.*

Proof: Let f, g and h be three acts such that $f \succ_{\Pi\Delta 0} g$ and $g \succ_{\Pi\Delta 0} h$. We are to prove that $f \succ_{\Pi\Delta 0} h$.



Notations: $\Delta(f,g) = A \cup B \cup C, \Delta(g,h) = A \cup B \cup D$ and $\Delta(f,h) = B \cup C \cup D$. We note also $F_X = \max_{s\in X} \min\{\mu(f(s)), \pi(s)\}$; $G_X = \max_{s\in X} \min\{\mu(g(s)), \pi(s)\}$ and $H_X = \max_{s\in X} \min\{\mu(h(s)), \pi(s)\}$.
Remark: $F_{X\cup Y} = \max\{F_X, F_Y\}$, $G_{X\cup Y} = \max\{G_X, G_Y\}$ and $H_{X\cup Y} = \max\{H_X, H_Y\}$.

$f \succ_{\Pi\Delta 0} g \Leftrightarrow F_{A\cup B\cup C} > G_{A\cup B\cup C} \Leftrightarrow \max\{F_A, F_{B\cup C}\} > \max\{G_{A\cup B}, G_C\}$ and $g \succ_{\Pi\Delta 0} h \Leftrightarrow G_{A\cup B\cup D} > H_{A\cup B\cup D} \Leftrightarrow \max\{G_{A\cup B}, G_D\} > \max\{H_A, H_{B\cup D}\}$. Notice that $G_D = F_D$ since $D \cap \Delta(f,g) = \emptyset$, $G_C = H_C$ since $C \cap \Delta(g,h) = \emptyset$ and $H_A = F_A$ since $A \cap \Delta(f,h) = \emptyset$. So, from $f \succ_{\Pi\Delta 0} g$ and $g \succ_{\Pi\Delta 0} h$ we get $\max\{F_A, F_{B\cup C}\} > \max\{G_{A\cup B}, H_C\}$ and $\max\{G_{A\cup D}, F_D\} > \max\{F_A, H_{B\cup D}\}$. Then, combining the two inequalities by a max: $\max\{\max\{F_A, G_{A\cup B}\}, F_{B\cup C\cup D}\} > \max\{\max\{F_A, G_{A\cup B}\}, H_{B\cup C\cup D}\}$. But, $\max\{F_A, G_{A\cup B}\} \leq \max\{\max\{F_A, G_{A\cup B}\}, H_{B\cup C\cup D}\}$, so $F_{B\cup C\cup D} > \max\{\max\{F_A, G_{A\cup B}\}, H_{B\cup C\cup D}\}$ and $F_{B\cup C\cup D} > H_{B\cup C\cup D}$, that is $f \succ_{\Pi\Delta 0} h$. □

Thus, we have proven that $\succ_{\Pi\Delta 0}$ is transitive. Let now α^* be the lowest non null value, that is: $\forall \alpha >_L 0$, $\alpha \geq_L \alpha^*$. Such an α^* exists since we assume that the scale L is finite. Then we prove that $\succ_{\Pi\Delta_{\alpha^*}}$ is transitive:

Proposition 13 *Let* α^* *be such that* $\forall \alpha >_L 0$, $\alpha \geq_L \alpha^*$. *Then* $\succ_{\Pi\Delta_{\alpha^*}}$ *is transitive.*

Proof: Let us consider the notations and the proof of prop. 12. It holds that:
(1) $f \succ_{\Pi\Delta_{\alpha^*}} g \Rightarrow \max\{F_A, F_{B\cup C}\} > \max\{G_{A\cup B}, G_C\}$,
(2) $g \succ_{\Pi\Delta_{\alpha^*}} h \Rightarrow \max\{G_{A\cup B}, G_D\} > \max\{H_A, H_{B\cup D}\}$.
From which we prove: $\max\{F_{B\cup C}, F_D\} > \max\{H_C, H_{B\cup D}\}$. But this means that either $f \succ_{\Pi\Delta_{\alpha^*}} h$ or both
(3) $\max\{F_{B\cup C}, F_D\} = \alpha^*$ and
(4) $\max\{H_C, H_{B\cup D}\} = 0$.

Then, we show that (1), (2), (3) and (4) are inconsistent together, from which we get the desired result that $f \succ_{\Pi\Delta_{\alpha^*}} h$. Notice that the 4 equations may be transformed equivalently in:
(1) $\max\{F_A, F_{B\cup C}\} > G_{A\cup B}$
(2) $\max\{G_{A\cup B}, F_D\} > F_A$
(3) $\max\{F_{B\cup C}, F_D\} = \alpha^*$ and
(4) $H_C = H_B = H_D = 0$.
Then, if we assume $F_A = 0$, since $A \subseteq \Delta(f,g)$ we have $G_A \geq \alpha^*$ and since from (3) $F_{B\cup C} \leq \alpha^*$, (1) can not be verified.
So, $F_A \geq \alpha^*$, but from (3), $F_D \leq \alpha^*$ then by (2), $G_{A\cup B} > F_A$, but now, since from (3) $F_{B\cup C} \leq \alpha^*$, (1) can not be verified.

We conclude that (1), (2), (3) and (4) are inconsistent together, from which we get the desired result that $f \succ_{\Pi\Delta_{\alpha^*}} h$. □

Unfortunately, $\succ_{\Pi\Delta_\alpha}$ is not transitive in general, as soon as there exists $\alpha', \alpha > \alpha' > 0$.

Proposition 14 *As soon as* $|S| \geq 2$ *and* $|X| \geq 3$, $\forall\alpha/\exists\alpha', \alpha > \alpha' > 0$, $\exists, \pi, \mu$ *such that* $\succ_{\Pi\Delta_\alpha}$ *is not transitive.*

Proof: Let $S = \{s_1, s_2\}$, $X = \{x_0, x_1, x_2\}$, $\pi(s_1) = \pi(s_2) = 1_L$, $\alpha \geq \mu(x_2) > \mu(x_1) > \mu(x_0)$ and $f(s_1) = h(s_1) = h(s_2) = x_0$, $g(s_1) = g(s_2) = x_1$ and $f(s_2) = x_2$. $f \succ_{\Pi\Delta 0} g$ since $\Delta(f,g) = S$ and $U_\Pi(f) = \mu(x_2) > U_\Pi(g) = \mu(x_1)$. $g \succ_{\Pi\Delta 0} h$ since $\Delta(g,h) = S$ and $U_\Pi(g) = \mu(x_1) > U_\Pi(h) = \mu(x_0)$. But, $\Delta = \Delta(f,h) = \{s_2\}$ and $U_\Pi(f\Delta\alpha) = \max\{\mu(f(s_2)), \alpha\} = \alpha$ and $U_\Pi(h\Delta\alpha) = \max\{\mu(h(s_2)), \alpha\} = \alpha$, so $f \sim_{\Pi\Delta 0} h$ that is $\succ_{\Pi\Delta_\alpha}$ is not transitive. □

4.2.3 Conclusions about the choice of α in $\succeq_{\Pi\Delta_\alpha}$ and $\succeq_{N\Delta_\alpha}$

As we have seen in the two preceding sections, choosing $\alpha = 0$ in $\succeq_{\Pi\Delta_\alpha}$ both guarantees that the strict part of the preference relation on acts is transitive and is the most refined among the family of $\succ_{\Pi\Delta_\alpha}$ partial orderings.

Every propositions in these two sections have their exact counterpart concerning $\succeq_{N\Delta_\alpha}$. In particular, one can show that $\alpha = 1$ is the best choice in this case, insofar as $\succ_{N\Delta 1}$ is both transitive, and the most refined among the family of $\succ_{N\Delta_\alpha}$ partial orderings.

Unfortunately, the $\succeq_{\Pi\Delta_\alpha}$ and $\succeq_{N\Delta_\alpha}$ are not complete preorders, and thus cannot be represented by a utility function. Nevertheless, they may be useful for choosing among actions that are ex-aequo according to the possibilistic decision criteria that they refine.

5 Concluding remarks

We have described a general family of decision procedures derived from the family of monotonic utilities, and that obey Savage's Sure Thing Principle. Unfortunately, these decision procedures possess, in general, only very few interesting properties.

Fortunately, in some special cases we recover interesting properties: In particular, among the decision procedures that are derived from the possibilistic utilities, some are refinements of the original decision procedures, and their strict part is transitive.

Now, this work could be naturally extended into two directions:

- First, we should search for an axiomatization of the discri-possibilistic decision procedures, in the style of Savage. For instance, we already now that these procedures obey the Sure Thing Principle and are no more representable by a complete preorder (although the strict preference relation is transitive). A careful study of their behavior in front of the axioms of expected utility and possibilistic utility should help in finding this axiomatization.
- Second, we have shown in this article that discri-monotonic decision procedures are not acyclic in general, and thus, very difficult to accept. When uncertainty is modeled by a qualitative possibility or necessity, acyclicity s recovered. Another direction for future research would then consist in determining those uncertainty models for which acyclicity is guaranteed. Good candidates are qualitative probabilities, qualitative beliefs and plausibilities [17].

References

[1] B. Bonet and H. Geffner. Arguing for decisions: A qualitative model of decision making. In F. Jensen eds. E. Horwitz, editor, *Proc. 12th Con. on Uncertainty in Artificial Intelligence (UAI'96)*, pages 98–105, Portland, Oregon, july 31-august 4 1996.

[2] C. Boutilier. Toward a logic for qualitative decision theory. In P. Torasso eds. J. Doyle, E. Sandewall, editor, *Proc. 4th Inter. Conf. on Principles of Knowledge Representation and Reasoning (KR'94)*, pages 75–86, Bonn, Germany, may 24-27 1994.

[3] R. Brafman and M. Tennenholtz. On the axiomatization of qualitative decision criteria. In *Proc. of the 14th National Conf. on Artificial Intelligence (AAAI'97)*, pages 76–81, Providence, R. I., Aug. 4-8 1997. AAAI Press / MIT Press.

[4] D. Dubois. Possibility theory and decomposable confidence measures on finite sets. *Computers and Artificial Intelligence*, 5:404–416, 1986.

[5] D. Dubois, H. Fargier, and H. Prade. Possibilistic likelihood relations. In *7th conf. on Information Processing and Management of Uncertainty (IPMU'98)*, pages 1196–1203, Paris, La Sorbonne, 1998. EDK.

[6] D. Dubois and H. Prade. *Possibility Theory*. Plenum Press, New York, N. Y., 1988.

[7] D. Dubois and H. Prade. Possibility theory as a basis for qualitative decision theory. In C. S. Mellish, editor, *Proc. of the 14th Int. Joint Conf. in Artif. Intelligence (IJCAI'95)*, pages 1925–1930, Montreal, Canada, Aug. 20-25 1995. Morgan Kaufman.

[8] D. Dubois, H. Prade, and R. Sabbadin. Towards axiomatic foundations for decision under qualitative uncertainty. In M. Mareš et al., editor, *Proc. of the 7th World Congress of the Inter. Fuzzy Systems Association (IFSA'97)*, pages 441–446, Prague, Czech Republic, June, 25-29 1997. Academia Verlag.

[9] D. Dubois, H. Prade, and R. Sabbadin. Qualitative decision theory with sugeno integrals. In G. F. Cooper and S. Moral, editors, *Actes de la 14eme Conf. Uncertainty in Artificial Intelligence (UAI'98)*, pages 121–128, Madison, WI, July 24-26 1998. Morgan Kaufmann.

[10] I. Gilboa. Expected utility with purely subjective non-additive probabilities. *J. of Mathematical Economics*, 16:65–88, 1987.

[11] J. L. Marichal. On Sugeno integrals as an aggregation function. Technical Report 9710, GEMME, Faculté d'Economie, de Gestion et de Sciences Sociales, Liège, Belgium, May 1997.

[12] J. Pearl. From conditionnal oughts to qualitative decision theory. In D. Heckerman and A. Mamdani, editors, *Proc. of the 9th Conf. on Uncertainty in Artificial Intelligence (UAI'93)*, pages 12–20, San Mateo, CA, 1993. Morgan Kaufmann.

[13] L. J. Savage. *The Foundations of Statistics*. J. Wiley and Sons, New York, 1954.

[14] D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, 57:571–587, 1989.

[15] M. Sugeno. Fuzzy measures and fuzzy integrals - A survey. In *Fuzzy Automata and Decision Processes*, pages 89–102. M. M. Gupta, G. N. Saridis et B. R. Gaines éditeurs, North-Holland, Amsterdam, 1977.

[16] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, N. J., 1944.

[17] S. K. M. Wong, Y. Y. Yao, and P. Lingras. Comparative beliefs and their measurements. *Int. Journal of General Systems*, 22:69–89, 1993.

[18] L. A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1:3–28, 1978.