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Invited Review

Fuzzy scheduling: Modelling flexible constraints vs. coping with incomplete knowledge

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Abstract

An overview of some fuzzy set-based approaches to scheduling is proposed, emphasizing two distinct uses of fuzzy sets: representing preference profiles and modelling uncertainty distributions. The first setting leads to a valued, non-compensatory generalization of constraint-directed scheduling. The other setting yields a possibility-theoretic counterpart of PERT, where probability distributions of activity durations are changed into possibility distributions, for the purpose of modelling incomplete information. It is pointed out that a special case of the latter, interval-valued PERT, is a difficult, ill-known problem, regarding the determination of critical activities, latest starting times and floats. Lastly when flexible constraints and uncertain processing times are to be jointly considered, the use of possibilistic decision theory leads to the computation of robust schedules.

Keywords: Scheduling; Constraint propagation; Uncertainty; Fuzzy intervals; Possibility theory

1. Introduction

Classical formulations of scheduling problems can be split into two trends: the optimisation of a single criterion, such as the makespan, for instance, and the constraint-directed approach under time and resource constraints. In the first approach, by far the most usual one, an interesting schedule is produced. But it is not always useful in practice because other down-to-earth criteria have been neglected. In the second approach, local specifications can be expressed, but there are two potential pitfalls: one is not to find any solution to the set of constraints despite extensive computation if the problem is overconstrained; the other difficulty is when there are too many solutions, so that the user, whose preferences have not been modelled, cannot easily decide. A third approach to scheduling uses priority rules (MacCarthy and Liu, 1993), but then it is difficult to understand what kind of criterion is at stake, and to what extent a solution is better than another, although the approach is computationally attractive. Moreover, scheduling is often stated as a deterministic problem and assumes precise knowledge of the data such as task durations, due-dates etc.

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There are stochastic versions of scheduling problems (Lootsma, 1989; Birge and Dempster, 1996), but they are hard to compute in practice, because some deterministic scheduling problems are already very hard.

Resorting to fuzzy set and possibility theory may help building a tradeoff between the expressive power and the computational difficulties of stochastic scheduling techniques while tackling uncertainty and accounting for local specifications of preferences. This kind of methodology is not yet so common in operational research, even if quite a few works in fuzzy PERT–CPM and other types of fuzzy scheduling methods have been around for more than two decades (Dubois and Prade, 1978; Prade, 1979). Overviews on various aspects of fuzzy scheduling can be found in the book by Lootsma (1997), a recent edited volume (Slowinski and Hapke, 2000) and papers by Chanas and Kuchta (1998) on graph-theoretic aspects, Werners and Weber (1999) on fuzzy project management and Turksen and Fazel Zarandi (1999) on fuzzy rule-based production management. An abundant bibliography on fuzzy set applications in production management is supplied in Guiffrida and Nagi (1998).

One difficulty with fuzzy scheduling is to figure out what problem is really addressed in the various works found in the literature. If we set aside the use of fuzzy sets in the modelling of priority rules applied to deterministic formulations, fuzzy scheduling addresses two very distinct issues: scheduling under flexible constraints and scheduling under incomplete or imprecise information. In the first group of papers, fuzzy sets are used to model local or global requirements in the form of flexible constraints (Zadeh, 1975; Dubois et al., 1996) and the problem is to find the best schedule that achieves a compromise between these requirements. This methodology is akin to constraint directed methods, and includes the optimisation of a single criterion as a particular case. In the second group of papers, the aim is to analyse the main characteristics of a scheduling problem (minimal makespan, critical paths, earliest and latest starting times of tasks, etc.) when data, especially task durations, are ill-known and modelled by fuzzy intervals, in the setting of possibility theory (Zadeh, 1978; Dubois and Prade, 1988). Possibility theory proposes a natural framework, simpler and less data-demanding than probability theory, for handling incomplete knowledge about scheduling data.

Since flexible constraints and uncertain data can be modelled by fuzzy sets, there is a risk of confusing the two purposes of fuzzy modelling in scheduling problems. Indeed, in the scope of decision theory, fuzzy sets can be used either as substitute of utility functions or as substitute of probability functions. It reflects the ambiguity of membership functions, that can be used both for preference modelling and for uncertainty handling. Yet, some scheduling problems involve both flexible constraints and uncertain data. Then, instead of optimizing average behaviors like in stochastic scheduling, fuzzy techniques rather aim at finding robust fault-tolerant schedules where all constraints are satisfied to some extent, with a sufficient level of confidence.

This paper is a structured discussion of the state of the art in the fuzzy scheduling literature with a stress on the distinction between preference modelling and uncertainty analysis using fuzzy set-based methods. The next section outlines the flexible constraint methodology in scheduling problems without, and then with, limited resources. Section 3 is devoted to the possibilistic uncertainty analysis of scheduling problems. It is shown that although many works exist along this line, a full-fledged critical path analysis under uncertainty has not been proposed yet. Nevertheless, very recent works come to grip with this matter. Lastly the hybrid problem of scheduling under incomplete information and flexible constraints is discussed.

2. Scheduling under flexible constraints

Using flexible constraints in scheduling (in production engineering, and also in project management) is a way of coping with the limitations of classical formal statements of scheduling problems. Finding the optimal schedule in the sense of a unique criteria does not account for the fact that very often, a schedule is better than another in practice because it better fulfills local requirements that are not modelled by the objective function. Constraint-directed approaches (Erschler et al., 1976; Erschler and Esquirol, 1986; Esquirol and Lopez, 1999; Fox, 1987) do account for the presence of local requirements. However they do it in a crude, all-or-nothing manner while optimisation approaches evaluate the worth of schedules in a more refined way. Using flexible constraints preserves the idea that the worth of a schedule is a matter of degree, while remaining faithful to the spirit of constraint-directed modelling where there is no compensation between the satisfaction of local antagonistic constraints. We successively deal with the case of unlimited resources constraints and then the case of limited resources.

2.1. The flexible constraint view of fuzzy PERT

Consider a set of activities (or tasks) related by precedence constraints expressing that some activities cannot start before the end of others. Each activity i is assigned a duration p_i. This partially ordered set models a project (or a job in the production engineering context). Usually, two fictitious tasks α and ω of null duration are added to this set, standing for the starting point and the ending point of the project, respectively: α precedes all activities, and ω takes place after all activities. When resource constraints are not taken into account, a project is thus modelled by a directed acyclic graph G whose nodes represent activities and arcs stand for precedence relations. Let $\mathrm{Succ}(i)$ (resp. $\mathrm{Pred}(i)$) denote the set of activities immediately following (resp. preceding) activity i, while $\mathrm{SUCC}(i)$ (resp. $\mathrm{PRED}(i)$) denote set of all activities taking place after (resp. before) activity i. Given two activities i and j in the graph, let $\boldsymbol{C}(i,j)$ denote the set of all paths from i to j, t_j^- and t_j^+ earliest and latest starting times of activity j. The float of activity j is $f_j = t_j^+ - t_j^-$. An activity is said to be critical whenever its float is zero.

The particular scheduling problem considered here is that of determining feasible values for the starting times t_i of activities, and possibly feasible values of the activity durations, under various constraints including some on the launching date t_α and the ending date t_ω of the project. It is indeed assumed that the project can be launched only after a prescribed release date r and must be finished before a prescribed due-date d. Hence the additional constraints:

$$\text{Relase date constraint}: t_\alpha \geqslant r, \tag{1}$$

$$\text{Due-date constraint}: t_\omega \leqslant d. \tag{2}$$

This simple constraint satisfaction problem may fail to have solutions, in terms of feasible starting times t_i and durations p_i. The earliest starting time of each activity i only depend on the durations of activities in $\mathrm{PRED}(i)$ and the release date, while the latest starting time of activity i is a function of the durations of activities in $\mathrm{SUCC}(i)$, the duration p_i, and the due-date. If the earliest starting time of some activity is larger than its latest starting time, it means that the problem is unfeasible, and these activities have negative floats. On the contrary, for some data sets, there are no critical tasks.

2.1.1. Modelling flexible requirements

If there are specific temporal constraints on some activities such as local release dates r_i (availability of raw material) or due-dates d_i (status review dates) there may be additional constraints under the form of local feasibility windows:

$$r_i \leqslant t_i \leqslant d_i - p_i. \tag{3}$$

In practice, such constraints are often flexible. The feasibility window $[r, d]$ in which a project much be carried out is made flexible if preferences of suppliers or customers are accounted for. A customer places an order with a given due-date, which is the preferred delivery date d^*; but a certain delay is tolerable up to a later date d beyond which the order will be canceled, because this customer will have settled the matter via

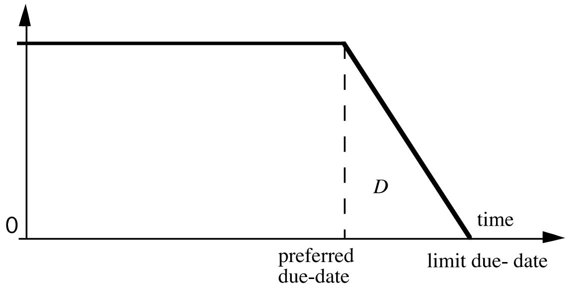

Fig. 1. Flexible due-date constraint.

some other supplier, for instance. As time passes between the preferred delivery date and the maximal due-date, the customer satisfaction decreases until it vanishes at the latter. The greater the delay, the lower the satisfaction. Such a decreasing preference profile is pictured on Fig. 1. It takes the form of a fuzzy set D with a decreasing membership function expressing a *flexible threshold* "less than". It should be obtained after some negotiation with the customer. This is the global flexible due-date, that results in a specific satisfaction level $\mathrm{sat}(t_\omega) = \mu_D(t_\omega)$ for the ending date of the project.

Supplier preferences for delivery dates that affect the starting of activities are modelled similarly. For suppliers, the later a delivery date, the better, so as to avoid tight schedules. A flexible release date constraint takes the form of a fuzzy set R with an increasing membership function expressing a flexible threshold "greater than". For instance, it results in a specific satisfaction level $\mathrm{sat}(t_\alpha) = \mu_{R,}(t_\alpha)$, for the starting date of the project. Clearly, $\mu_R(t_\alpha) = 0$ if $t_\alpha \leqslant r$ (impossible to start before r), and $\mu_R(t_\alpha) = 1$ if $t_\alpha \geqslant r^*$ (more sensible to start after r^*). Fuzzy release date and due-date constraints form a flexible time horizon where the project is bound to take place.

Besides, durations of activities are sometimes controlable, hence are a matter of preference as well. For instance, tuning the speed of a machine-tool may affect the duration of a machining operation; setting the temperature of an oven affects the duration of a drying operation; assigning more or less staff to a man-made activity also changes the activity duration. The basic local criterion involved here is the quality of the result obtained by the activity, the comfort or the safety of the people involved in it. For each activity, there may exist optimal values of the duration that achieve such goals, and some values that are less recommended (not quite within strict quality or safety bounds). The preference profile pertaining to an activity duration can be modelled by a fuzzy interval (see Appendix A) P_i: a choice of the duration p_i for activity i is fully satisfactory if it lies in the core of P_i containing the best values $(\mu_{P_i}(p_i) = 1)$. Such a choice is forbidden if it is outside the support of $P_i(\mu_{P_i}(p_i) = 0)$. The closer to the core of P_i is the choice of p_i, the better. Each such specification on the duration leads to a local flexible constraint with satisfaction level: $\mathrm{sat}(p_i) = \mu_{P_i}(p_i)$.

Generally, the useful part of the fuzzy interval P_i is its increasing side, since the other one is not conflicting with the fuzzy realizability windows. It expresses the following kind of requirement: the longer the duration, the better the quality (but the more likely it conflicts with other temporal constraints). Everything being equal otherwise, the lowest value of processing time should be chosen, as it allows for larger slack times in the schedule, hence for better satisfaction of the ready date and the due-date. In this Section, we shall assume that membership function of processing times have increasing membership functions expressing flexible thresholds "greater than".

To summarize, various kinds of requirements can be modelled for the choice of starting times and durations of activities. The problem is to find a schedule that best satisfies all above requirements coming from customers, suppliers and the project manager.

2.1.2. Solving the flexibly constrained scheduling problem

In this context, a potential solution to the problem is a choice $s = (t_\alpha, \ldots, t_\omega, p_\alpha, \ldots, p_\omega)$ of starting times and processing times for activities. They must satisfy hard precedence constraints expressed by the graph, and flexible constraints induced by release dates and due-dates described above. In accordance with the constraint-directed view, the problem is stated as one of maximin optimisation (Bellman and Zadeh, 1970): the degree of satisfaction of potential solution s is the degree of satisfaction of the least satisfied constraint: $\mathrm{sat}(s) = 0$ if $s = (t_\alpha, \ldots, t_\omega, p_\alpha, \ldots, p_\omega)$ violates a single precedence constraint. Otherwise:

$$\mathrm{sat}(s) = \min(\min_i \mu_{P_i}(p_i), \mu_D(t_\omega), \mu_R(t_\alpha)). \tag{4}$$

Local fuzzy feasibility windows defined by local constraints R_i and D_i on release dates and due-dates for activities could be accounted for, but are not here for simplicity. The problem to maximize $\mathrm{sat}(s)$. This is a particular kind of fuzzy linear programming problem in the sense of Zimmermann (1985) that can be solved via fuzzy constraint propagation techniques (Dubois et al., 1995). The consistency degree $cons = \sup_s \mathrm{sat}(s)$ of the problem evaluates the level of feasibility of the scheduling problem. When constraints are partially incompatible, $0 < cons < 1$, because no choice of s can fully satisfy all the constraints. Then, a kind of automatic relaxation of constraints is performed aiming at finding values of starting times or processing times that remain in the vicinity of the unreachable ideal values.

This formulation of the flexible scheduling problem tends to balance the levels of satisfaction across constraints for the sake of not violating any. It contrasts with additive approaches, maximizing the sum of local satisfaction levels (Sadeh and Fox, 1996). In this case degrees of membership would model rewards (depending on costs). Maximizing the sum of membership grades may produce optimal schedules where $\mu_{P_i}(p_i) = 0$ for some activity i, or whose launching date or release date is unfeasible, which contradicts the spirit of constraint-directed solving.

For solving the problem in practice, it is possible to come down to finding choices of starting times $t = (t_\alpha, \ldots, t_\omega)$ first, using a constraint propagation step that gets rid processing times. The choice of a vector of starting times t is feasible provided that there exists a choice $p = (p_\alpha, \ldots, p_\omega)$ of processing times such that the solution $s = (p, t)$ is optimal:

$$\mathrm{sat}(t) = \sup_p \mathrm{sat}(t, p) = cons. \tag{5}$$

It can be shown that (Dubois et al., 1995)

$$\mathrm{sat}(t) = \min_{(j,k)\ \text{such that}\ j \in \mathrm{Pred}(k)} \min(\mu_{P_j}(t_k - t_j), \mu_D(t_\omega), \mu_{R,}(t_\alpha)). \tag{6}$$

It comes down to a scheduling problem involving only starting times, where each precedence constraint has become flexible:

$$(t_j - t_i) \geqslant P_i \tag{7}$$

with satisfaction degree $\mu_{P_i}(t_j - t_i)$. The fuzzy constraint propagation approach enables a fuzzy set of more or less feasible starting times to be computed rather than just the optimal ones, which is more valuable for the user. The starting time t_i of activity i is constrained by the earliest ending times of activities in $\mathrm{PRED}(i)$ and the latest starting times of activities in $\mathrm{SUCC}(i)$. Earliest ending times in $\mathrm{PRED}(i)$ and latest starting times in $\mathrm{SUCC}(i)$ are independent quantities here. These parameters are modelled by fuzzy earliest starting times T_i^- and latest starting times T_i^+. They can be computed by a straightforward extension of the standard forward and backward recursive schemes, *performed independently*, using fuzzy additions and subtractions and the fuzzy extensions of the minimum and the maximum operations (see Appendix A):

$$T_i^- = \widetilde{\max}_{j \in \mathrm{Pred}(i)} T_j^- \oplus P_j \quad \text{for } i > \alpha, \text{ and } T_\alpha^- = R, \tag{8}$$

$$T_i^+ = \widetilde{\min}_{j \in \mathrm{Succ}(i)} T_j^+ \Theta P_i \quad \text{for } i < \omega, \text{ and } T_\omega^+ = D. \tag{9}$$

If available, independent local fuzzy constraints on release dates R_i and due-dates D_i activities could be accounted for in (8) and (9), respectively. The obtained membership functions of T_i^- and T_i^+ are respectively increasing and decreasing. Then the local satisfaction degree $\mathrm{sat}(t_i)$ of a choice of the starting time T_i of activity i reads:

$$\mathrm{sat}(t_i) = \sup_{t_j: j \neq i} \mathrm{sat}(t_\alpha, \ldots, t_\omega) = \min(\mu^+_{T_i}(t_i), \mu^-_{T_i}(t_i)). \tag{10}$$

The set of more or less preferred values of the starting time t_i of activity i is the fuzzy set $T_i^- \cap T_i^+$, and $\mathrm{sat}(t_i)$ is the height of this fuzzy set. The overall consistency level can computed at the end of the forward propagation, for instance: it is the height of the fuzzy set $T_\omega^- \cap D$, from which the preferred ending date of the project is obtained.

If $0 < cons < 1$, precise optimal values of starting times $t_i = \mu^{-1}_{T_i^-}(cons)$ or processing times $p_i = \mu^{-1}_{P_i}(cons)$ can be computed (if $cons = 1$, intervals are obtained). Processing times are obtained by inverting the membership function of P_i's at level $cons$. Earliest starting times are similarly obtained from T_i^-'s, thus yielding a deterministic PERT network. Nevertheless, if the so-computed schedule is indeed optimal in the sense of the "bottleneck" criterion (4) it may fail to be Pareto-optimal in the sense of the vector optimisation problem based on the set of membership functions that define the various flexible constraints. Indeed, only the starting times and processing times of critical activities (those s.t. $t_i + p_i = \mu^{-1}_{T_i^+}(cons)$) cannot be improved. They form a set of critical paths in the defuzzified nework, in the usual sense. It is possible to lengthen the processing times of non-critical tasks thus increasing their membership values $\mu_{P_i}(p_i)$. To this end, another flexible scheduling problem can be solved where all durations of critical activities are fixed to $p_i = \mu^{-1}_{P_i}(cons)$, and the feasibility window of the project is set to $[r, d] = [\mu^{-1}_R(cons), \mu^{-1}_D(cons)]$. If $cons < 1$, a unique Pareto-optimal solution can thus be calculated by proceeding in a recursive way until all tasks are (artificially) made critical (see Dubois and Fortemps, 1999). More general kinds of fuzzy scheduling problems are described by Wang and Fu (1996). Their aim is to minimize costs under flexible constraints on activity times or available budget. The adopted methodology fuzzy linear programming.

2.2. Flexible constraint-based scheduling under limited resources

The main difficulty of scheduling problems lies in the necessity of performing activities under limited resources. Here we only consider the case of renewable resources, that are again fully available at the end of the activity that used them. Basically resources are machines, tools, or human operators. We focus on the case of disjunctive constraints which prevent two activities from using the same resource simultaneously, without prescribing any precedence between them.

The flexible constraint methodology straightforwardly extends to resource-constrained scheduling problems, such as flow-shops and job-shops. Then resources are precisely machines, each activity requiring a machine during the time when it is performed. The basic idea for solving such scheduling problems under flexible constraints is to solve all fuzzy scheduling problems obtained by sequencing the set of tasks assigned to each machine. Each sequencing leads to an optimal choice of starting times and processing times of activities, in the sense of the maximin criterion, and the best sequencing can be laid bare. Methods differ by the way the set of possible sequencings is explored. Ishii et al. (1992) consider one-machine scheduling problems with fuzzy due-dates. It is extended to the case of flow-shop problems by Ishibuchi et al. (1994) who use various metaheuristics. The case of jobshop scheduling under fuzzy constraints on due-dates, release dates and processing times is considered by Dubois et al. (1995). The solving method is an enumeration procedure based on a sequence of fuzzy constraint propagation steps that update fuzzy earliest starting time and latest starting time of activities, and decision steps that solve disjunctive constraints, using fuzzy extensions of look-ahead procedures coming from constraint-directed scheduling methods (Erschler et al., 1976). For instance, each pair of activities on a machine is tested to see if they both fit in the current (fuzzy) window allocated to them. This test often enforces a sequencing of these two activities, or, in the fuzzy case, suggests it more or less strongly. Other applications of fuzzy constraints to scheduling include Litoiu and Tadei (1997) (real-time periodic scheduling), Hintz and Zimmermann (1989) (master scheduling in flexible manufacturing systems), Slany (1996) (scheduling using a generic fuzzy constraint-directed problem solver). Fargier (1997) reports on systematic experiments in jobshop scheduling with a fuzzily constrained makespan, and shows that the use of fuzzy constraints generally accelerates the discovery of a feasible solution, as opposed to the hard constraint versions of the same problems, and that this solution is found in the middle of the feasibility domain, while the hard version tends to find solutions on the borderline of the feasibility domain.

In contrast with the fuzzy constraint-directed approach, several authors have used fuzzy sets at the control level, in the modelling of priority rules, applied to deterministic scheduling problems. This topic is out of the scope of this overview, but deserves some comments. Fuzzy priority rules have two merits: first, as usual with fuzzy rules, threshold effects are avoided; second, blending priority rules is made easier. Degrees of applicability of various rules to a situation can be computed, scaled, and combined with various weights in order rank sequencing decisions. Since the effect of individual priority rules on the attainment of various production goals is not so clear, the weighting and blending of fuzzy priority rules are far from being trivial problems. Bensana et al. (1988) combine a constraint-directed approach and a set of fuzzy priority rules whose results are aggregated by means of a majority technique, to solve jobshop scheduling problems. Voting methods for priority rule aggregations are explored by Dubois and Koning (1994). The empirical determination weights in the blending of priority rules is studied by Grabot and Geneste (1994) using neural network algorithms. They show that weights depend on the amount of resources, of the variance of activity durations, and the objective which is aimed at. The use of approximate reasoning methods in the style of fuzzy logic controllers, applied to various kinds of scheduling problems has been extensively studied by Turksen. See the survey in Turksen and Fazel Zarandi (1999).

3. Fuzzy scheduling with ill-known processing times

A very different use of fuzzy sets in scheduling problems is when activity durations ill-known at the moment when a predictive schedule must be devised, and that this lack of knowledge must be dealt with as such. It may also be the case that processing times have unpredictable variability. For instance, this is true for subcontracted activities in manufacturing environment, for debugging tasks in software engineering, and for activities to which resources have not been assigned to yet. Then, processing times are no longer considered as decision variables, but the problem is to cope with the uncertainty pervading them. Modelling an ill-known processing time by a probability distribution presupposes much knowledge (for instance statistical), or a devoted Bayesian decision-maker. In most cases, there is little knowledge available, and the crudest representation is a human-originated confidence interval which is supposed to contain the eventually observed value of a processing time, with sufficient certainty. In the case, when some values appear more plausible than others, the natural extension of an interval is a fuzzy interval (Dubois and Prade, 1980, 1988; Dubois et al., 2000), that is, a possibility distribution representing more or less plausible values, viewed as a nested family of confidence intervals (see Appendix A).

Two kinds of scheduling problems have been addressed in this setting: the minimization of the makespan, and the determination of a robust schedule under fuzzy constraints. We first consider the case of makespan minimization with unlimited resources.

3.1. Critical path analysis with ill-known processing times

Strangely enough, the PERT analysis with ill-known processing times modelled by simple intervals does not seem to have received much attention in the literature. Yet, the predictive computation of the minimal completion time of a project, the determination of critical paths and activities, the determination of activity floats have been considered as important problems and have been widely acknowledged to be pervaded with uncertainty. However the overwhelming part of the literature devoted to this topic adopts an orthodox stochastic approach, thus leading to a very complex problem that is still partially unsolved to-date. Until recently, and to the best of the authors' knowledge, interval-valued PERT analysis seems to have existed only as a special case of fuzzy PERT studies that appeared in the late seventies. However, as seen below, the main difficulty of the criticality analysis problem in fuzzy PERT, when fuzzy intervals represent ill-known processing times, does not lie in introduction of fuzzy sets. It is already present when only usual intervals are involved. Solving the interval valued case is the main difficulty. The fuzzy case can then be rather easily solved, via the use of level-cuts.

Modelling ill-known processing times with fuzzy numbers is rather simple. It is justified by the limited expressive power of intervals: if too small, there is little confidence in them, too large, they are not informative enough to be exploited. Rommelfanger (1990) suggests the use of three confidence intervals: the core, containing typical values, the 0.5 level cut, containing unsurprizing values, and the support, outside which values are physically unattainable. It provides six parameters and the fuzzy interval is obtained by means of linear interpolation. A simpler model is the triangular fuzzy number using an interval and a plausible value in it.

The past fuzzy PERT literature has sometimes relied on the assumption that, since in deterministic PERT, most parameters of interest are obtained by means of simple algorithms involving addition, subtraction, minimum and maximum, the same algorithms, once fuzzified, can be straighforwardly used: the same calculations can be carried out, changing numbers into fuzzy numbers, exploiting results in fuzzy arithmetics (as done in the previous section in presence of fuzzy constraints).

The first interesting problem is that of computing the fuzzy completion time of the project. It is the possibility distribution of the minimal completion time. There is no constraint on the release date nor on the ending date. This question, and the related one of finding the shortest or the longest distance between nodes in a graph with fuzzy-valued arcs, is easy (contrary to the same question in stochastic PERT) and has been solved for a long time (Chanas and Kamburowski, 1981; Dubois and Prade, 1978, 1980; Gazdik, 1983; Mares, 1989). It is then assumed that the project starts at time 0, so that

$$t_\alpha^- = 0 \tag{11}$$

Let $\boldsymbol{C}(\alpha, i)$ be the set of paths (activity sequences) from the initial task to activity i. When processing times are precisely known the earliest starting time of activity i is the maximal length $L(\alpha, i)$ of paths in $\boldsymbol{C}(\alpha, i)$, supposing that arcs (i,j) are assigned length p_i. The minimal completion time is the maximal length $L(\alpha, \omega)$ of paths C in $\boldsymbol{C}(\alpha, \omega)$. When processing times are fuzzy, the length of path C is easily defined by adding the fuzzy numbers representing the processing times of activities in C, and applying the extended maximum:

$$L(C) = \oplus_{i \in C} P_i \quad \text{and } L(\alpha, \omega) = \widetilde{\max}_{C \in \boldsymbol{C}(\alpha, \omega)} L(C).$$

The possibility distribution of the minimal completion time is $T_\omega^- = L(\alpha, \omega)$ since the last activity has a zero duration. Fuzzy earliest starting times T_i^- of activities can actually be computed using the same forward propagation step as in the flexibly constrained PERT problem of Section 2.2. It is done by applying the recursion equation (8) until the last activity is reached, using fuzzy number arithmetics. The main difference is that the initial step is prescribed by (11) and that the fuzzy numbers P_i representing ill-known processing times are bell-shaped fuzzy intervals, instead of being fuzzy thresholds. This step by step procedure is correct because the earliest starting time of tasks is an increasing function of the processing times.

However a difficulty arises when it comes to checking for critical activities, computing latest starting times and floats. In the deterministic case, the backward recursion specified by Eq. (9) is carried out to compute latest starting times of activities, and initialized by

$$t_\omega^+ = t_\omega^- \tag{12}$$

since for criticality analysis, it is assumed that the project has minimal duration. Then floats can be computed and are never negative. Floats of activities represent the minimal temporal shift of starting times which do not alter the completion time of the project. Critical activities form one or several paths in $\boldsymbol{C}(\alpha, \omega)$.

When durations p_i are ill-known, it is tempting to compute the fuzzy latest starting times T_i^+ using the backward recursion method (9), intializing the process as

$$T_\omega^+ = T_\omega^- \tag{13}$$

and using fuzzy subtraction (Prade, 1979; MacCahon and Lee, 1988). However, as pointed out by several authors (Dubois, 1983; Nasution, 1993; Rommelfanger, 1994; Hapke et al., 1994), this method does not work for reasoning under uncertainty. The first mistake is to assume that the equality $T_\omega^+ = T_\omega^-$ does constrain the earliest starting time of the last task to be equal to its latest starting time. It is indeed not equivalent to enforcing $t_\omega^+ = t_\omega^-$ since two distinct variables may have the same distribution. The second mistake is to use fuzzy subtraction on fuzzy numbers that are actually interactive. Namely, it is actually wrong that $T_i^+ = \widetilde{\min}_{j \in \mathrm{Succ}(i)} T_j^+ \Theta P_i$ because the latest starting time t_j^+ already depends on the value of p_i since as soon as the constraint $t_\omega^+ = t_\omega^-$ is enforced, t_ω^+ depends on variables p_i. So, the computed values of latest starting times will be too imprecise, and the determination of critical activities cannot be made by comparing T_i^+ and T_i^-. For instance, if the project has a single task of duration $p \in [1, 4]$, the completion time distribution is $t_\omega^- = t_1^- + p \in [1, 4]$ too, and the latest starting time distribution computed by the procedure is $[1, 4]\Theta[1, 4] = [-3, +3] \neq 0$. However, this task is surely critical since $t_1^+ = t_1^- = t_\omega^- - p = 0$.

MacCahon (1993) proposes to go back to standard critical path methods via defuzzification of the fuzzy processing times. Kaufmann and Gupta (1988), Hapke et al. (1994) and Rommelfanger (1994) suggest substitutes to the fuzzy subtraction, so as to improve the situation, but these techniques remain ad hoc. The computation of distributions of latest starting times of activities cannot be achieved using elementary techniques of fuzzy arithmetics, not even of interval arithmetics in the non-fuzzy case. It is clear that the difficulty stems for the presence of intervals, be they fuzzy or not. Nasution (1993) resorts to symbolic computations on the variable processing times. However this technique is unwieldy and highly combinatorial.

Chanas and Kamburowski (1981) try and compute a criticality index for path and activities directly. One idea for checking if a path is critical may be to compute the height of the intersection of the fuzzy length $L(C)$ of a path C in $\boldsymbol{C}(\alpha, \omega)$ and the fuzzy completion time T_ω^-. It is clear that if $L(C) \cap T_\omega^-$ is empty then C is not critical. However the height of $L(C) \cap T_\omega^-$ may be 1 for paths that are surely not critical. For instance suppose three activities *A1, A2, A3* where *A1* precedes the two other ones. Assume that $P_1 = [1, 4]$, $p_2 = 2$, $p_3 = 3$. Then $T_\omega^- = [4, 7]$ and $C = (A1, A2)$ has imprecise length $[3, 6]$ but it is clearly never critical whatever the duration of task $A1$. The criticality of an activity i is similarly computed by comparing the fuzzy distribution of the maximal length L_i of paths in $C(\alpha, w)$ crossing task i, and the fuzzy completion time T_ω^-. L_i can be computed as $T_i^- \oplus \tau_i^-$ where T_i^- is the distribution of the maximal length of paths in $\boldsymbol{C}(i, \omega)$. This index clearly suffers from the same defect.

MacCahon and Lee (1988) propose to compute fuzzy slack times of activities as $T_i^+ \Theta T_i^-$ obtained by the forward and backward recursions, but the fuzzy variables restricted by T_i^+ and T_i^- are interactive so that what is obtained is only a rough imprecise approximation of the fuzzy range of the actual float of the activity. Such a computation makes sense only if the fuzzy due-date and the fuzzy release date of the projects are prescribed independently of each other (Dubois, 1983). Another view of criticality of activities could be based on the notion of "most vital arcs" in fuzzy graph problems (Lin and Chern, 1993): the idea would be to delete each activity in the network and see how it affects the fuzzy duration of the project. The most critical task could then be the one that maximally decreases the project length (using a fuzzy number ranking method).

Actually, a correct solution to the whole problem of critical path analysis under fuzzy uncertainty cannot be reached by mending existing algorithms. It requires a mathematically clean statement of the problem in the setting of possibility theory. This step was taken by Buckley (1989). Given a PERT graph with n activities, a tuple of n processing times $\Omega = (a_1, \ldots a_n)$ is called a *configuration*. Ω characterizes a regular PERT graph where $p_1 = a_1, p_2 = a_2, \ldots$ and $p_n = a_n$. In a given configuration Ω, $p_i(\Omega)$ denotes the value of p_i. In such a configuration, the earliest starting time $t_i^-(\Omega)$, the latest starting time $t_i^+(\Omega)$ and float $f_i(\Omega)$ of activity i can be computed.

The degree of possibility that p_i equals a prescribed value a is $\Pi(p_i = a) = \mu_{P_i}(a)$.

Insofar as processing times of the various activities are not related to one another, the degree of possibility of a configuration $\Omega = (a_1, \ldots a_n)$ is obtained by using the joint possibility distribution:

$$\pi(\Omega) = \Pi(p_1 = a_1 \text{ and} \ldots \text{and } p_n = a_n)) = \min_{i=1,\ldots n} \Pi(p_i = a_i) = \min_{i=1,\ldots n} \mu_{P_i}(a_i). \tag{14}$$

On this basis, the calculation of the distribution of any relevant parameter, such as the latest starting time or the float of some activity can be rigorously defined by projecting this joint distribution on the domain of this parameter. Criticality indices can also be rigorously defined as the degree of possibility and necessity (referring to the above joint possibility distribution) that a path or an activity is critical in the usual sense, the latter being viewed as events in the usual sense.

It is useful to state the problem when the available knowledge about processing times is given by intervals $[p_{*i}, p_i^*]$, as this is the core problem while the fuzzy problem is extension thereof. Then the possibility distribution π takes the form of an hyperparallelipiped H of dimension n if there are n activities (other than α and ω). To our knowledge, a fullfledged criticality analysis of interval-valued activity networks has never been published as topic of its own, although it sounds more realistic than deterministic models of projects. Actually, it is a partially open problem. Let us define the parameters of interest in a rigorous way.

Definition (*Criticality of a path*). A path C in $\boldsymbol{C}(\alpha, \omega)$ is said to be *possibly critical* (denoted PC) if and only if there exists a configuration Ω in H where path C is critical. This path C *is surely* (*necessarily*) *critical* (denoted SC) if and only if it is critical for all configurations Ω in H. This path is *surely non-critical* (or impossibly critical, denoted SNC) if and only if it is critical in no configuration Ω in H.

It is rather obvious that checking if a path C is PC is easy: a characteristic property is that C is critical in the configuration where processing times of activities on C take their maximal values, while they take their minimal values for other activities. Otherwise, C is SNC. Similarly, a path is SC if and only if it is critical in the dual configuration, where processing times of activities on C take their minimal values, while they take their maximal values for other activities. In the case of non-degenerated intervals for all processing times, an SC path is either unique or does not exist (Chanas et al., 2002). However there may be many PC paths. For activities, the definitions for criticality are just the same.

Definition (*Criticality of an activity*). An activity i is is said to be *possibly critical* (denoted PC) if and only if there exists a configuration Ω in H where activity C is critical. This activity i *is surely* (*necessarily*) *critical* (denoted SC) if and only if it is critical in configurations Ω in H. This activity is *surely non-critical* (or impossibly critical), denoted SNC) if and only if it is critical in no configuration Ω in H.

An activity is SC (resp. SNC) if and only if it belongs to all (resp. no) PC paths. An activity is PC if it is not SNC, hence belongs to at least one PC path. As a consequence there may be cases where no NC critical path exist, but where (isolated) NC activities can nevertheless be found. For instance, activities α and ω are always NC. Calculating these NC activities, as well as PC activities, turns out to be computationally difficult; see Chanas and Zielinski (2002) for PC activities and paths, and Chanas et al. (2002) for SC activities.

It is easy to check that only so-called *extreme configurations* (Ω such that $p_i = p_{*i}$ or p_i^*) need be used in the above definitions of criticality and there are 2^n such configurations, hence possible combinatorial complexity.

Now, the definition of earliest starting times, latest starting times and floats of activities in a given configuration Ω, in terms of path lengths $L^\Omega(C)$, is:

$$t_i^-(\Omega) = \max_{C\in\boldsymbol{C}(\alpha,i)} L^\Omega(C),$$

$$t_i^+(\Omega) = \max_{C\in\boldsymbol{C}(\alpha,\omega)} L^\Omega(C) - \max_{C\in\boldsymbol{C}(i,\omega)} L^\Omega(C),$$

$$f_i(\Omega) = \max_{C\in\boldsymbol{C}(\alpha,\omega)} L^\Omega(C) - \max_{C\in\boldsymbol{C}(\alpha,i)} L^\Omega(C) - \max_{C\in\boldsymbol{C}(i,\omega)} L^\Omega(C).$$

Hence, the definition of the ranges T_i^-, T_i^+, Φ_i of earliest starting times, latest starting times and floats of activities for interval-valued processing times is obvious:

$$T_i^- = \{t_i^-(\Omega), \Omega \in H\}; \quad T_i^+ = \{t_i^+(\Omega), \Omega \in H\}; \quad \Phi_i = \{f_i(\Omega), \Omega \in H\}.$$

The actual computation of these imprecise quantities is studied in Fargier et al. (2000), Dubois et al. (2001). It can be proved that these evaluations yield intervals that can be computed using extreme configurations only. The calculation (done above) of the earliest starting times is straightforward because $t_i^-(\Omega)$ is a monotonically increasing function of Ω (in the wide sense). However the other quantities $t_i^+(\Omega)$ and $\tau_i(\Omega)$ are not monotonic in all processing times. The precise behavior of these functions in H is not so simple to predict, which makes these parameters difficult to compute, as first noticed by Buckley (1989). More precisely, Dubois et al. (2001) notice that $t_i^+(\Omega)$ is monotonically increasing with respect processing times of activities $j \notin \text{SUCC}(i)$, and $j \neq i$, while $f_i(\Omega)$ is monotonically increasing with respect to processing times of activities $j \notin \text{SUCC}(i)$, $j \notin \text{PRED}(i)$, and $j \neq i$. Again, the computational complexity of procedures computing these quantities is potentially exponential. Fargier et al. (2000) provide optimal configurations where the least upper and the greatest lower bounds of T_i^+, Φ_i are attained, for networks having a series–parallel topology.

The study of the precise relationships between the earliest starting times, latest starting times and floats of activities, and the possible, sure, and impossible criticality notions defined above must still be carried out for interval-valued activity networks. Generally, the range of the float cannot be directly computed from the ranges of the earliest starting time and the latest starting time. However it is clear that when the float interval of a task is the singleton $\{0\}$, then the activity is SC; if the lower bound of this interval is positive, the activity is SNC. The situation in the interval-valued case thus considerably differs from the deterministic (completely informed case).

These notions can be straightfowardly extended to the fuzzy case, where processing times are fuzzy intervals and the membership function of the set of possible configurations is the possibility distribution π. The fuzzy framework leads to rigorous definitions of degrees of possible, sure and surely impossible criticality of paths and activities.

The degree of possible criticality of path C in $\boldsymbol{C}(\alpha,\omega)$ is the degree of possibility of finding a configuration where C is critical:

$$\text{PC}(C) = \sup_{\Omega:C \text{ critical in } \Omega} \pi(\Omega). \tag{15}$$

Now the companion quantity $\text{PNC}(C) = \sup_{\Omega:C \text{ not critical in } \Omega} \pi(\Omega)$, computing the possibility of the contrary event is closely related to the degree of sure criticality $\text{SC}(C)$ using the duality between necessity and possibility in possibility theory. Namely, the degree of sure criticality of path C is:

$$\mathrm{SC}(C) = 1 - \mathrm{PNC}(C) = 1 - \sup_{\Omega: C \text{ not critical in } \Omega} \pi(\Omega),$$

while $\mathrm{SNC}(C) = 1 - \mathrm{PC}(C)$, for the index of sure non-criticality of C. These notions can be expressed in terms of level-cuts $(P_i)_\alpha$ of fuzzy processing times P_i, so as to reduce the computations to the interval-valued case. Let G_α denote the interval-valued activity network with processing times $(P_i)_\alpha$:

$$\mathrm{PC}(C) = \sup\{\alpha : C \text{ is PC in } G_\alpha\}.$$
$$\mathrm{SC}(C) = 1 - \sup\{\alpha : C \text{ is PNC in } G_\alpha\}.$$

Criticality indices for activities can be defined similarly. The degree of possible criticality of activity i is the maximal degree of possible criticality for paths in $C(\alpha, \omega)$ containing i. The degree of sure criticality of activity i is λ if the maximal degree of possible non-criticality for paths in $C(\alpha, \omega)$ containing i is $1 - \lambda$. They are usually not so easy to compute. Chanas and Zielinski (2001) give some results for the computation of degrees of possible criticality, and Chanas et al. (2002) for indices of sure criticality.

Possibility distributions of latest starting times and floats of activities can be defined by means of the extension principle applied to their path-based expressions given above, using the joint possibility distribution of processing times π. See Fargier and Galvagnon (1999), Fargier et al. (2000), Dubois et al. (2003) where preliminary work for computing fuzzy latest starting times and fuzzy floats is described, especially in the case of series–parallel graphs. Again, a rigorous and complete criticality analysis of fuzzy PERT networks where fuzzy sets model ill-known processing times is still to be carried out, despite first results mentioned above.

3.2. Optimal scheduling with limited resources when processing times are ill-known

Solving a scheduling problem under limited resources, when the goal is to minimize the makespan, comes down to solving precedence conflicts on resources in the best way so as minimize the overall duration of the set of corresponding activities. Once precedence conflicts are solved in some way, the problem comes down to a fuzzy PERT like in the previous section, which is very simple in the deterministic case. More generally other scheduling criteria can be used, even leading to a multicriteria scheduling problem (Blazewicz et al., 1986). The literature is replete with techniques that solve deterministic scheduling problems under limited resources. The most efficient ones combine disjunctive graph representations with metaheuristics for optimisation (simulated annealing, noticeably) (see van Laarhoven et al., 1992; Fortemps and Hapke, 1997 for instance). Such methods yield good approximate solutions within reasonable time bounds, also in the fuzzy case. However the deterministic scheduling problems under limited resources remain computationally difficult.

In the interval-valued case, a fortiori in the fuzzy case, the unlimited resource scheduling problem sounds already computationally non-trivial, but the fuzzy minimal duration is easy to get. It sounds more reasonable to try and find a solution quickly, rather that looking for an optimal solution, all the more so when data are ill-known. The idea is then to change fuzzy data into precise data via defuzzification, find solutions using existing deterministic methods and compare solutions using fuzzy number comparison techniques (see Appendix A). This kind of idea is at work in (Fortemps, 1997) for jobshop problems with fuzzy durations. More specific problems like flow-shop are considered by MacCahon and Lee (1992). Özelkan and Duckstein (1999) test the optimality of classical priority rules in the fuzzy setting, for very specific scheduling problems, using defuzzified values of processing times. A more general setting for fuzzy scheduling in uncertain environments and limited resources is the one of Hapke et al. (1994) where renewable resources are also ill-known (like manpower due to absent people). Mares (1989) even envisages fuzzy sets of activities and fuzzy sets of preference constraints, to express uncertainty on the occurrence of activities and their ordering.

3.3. Scheduling under flexible constraints and ill-known processing times

Rather that minimizing the makespan, one may be more interested to find a schedule that satisfies local constraints like in Section 2. When there are flexible constraints on temporal parameters such as starting times or ending times of activities, and processing times are ill-known, the problem is one of flexibly constrained scheduling under uncertainty. The aim is to find starting times of activities so as to maximize the simultaneous satisfaction of constraints, whatever the actual processing times will turn out to be, within the limits fixed by our knowledge in the form of fuzzy intervals. Since the uncertainty is non-probabilistic, usual criteria such as the expected level of satisfaction of constraints do not apply. However, possibilistic decision rules under uncertainty exist and can be applied here (see Dubois et al., 2001). Especially the pessimistic decision rule (see Appendix A) may provide robust schedules.

In the non-fuzzy case the formulation of the problem sheds light on the type of solution it leads to and the method of getting it. Suppose hard constraints $R_i = \textit{at least } r_i$ on the release dates of activities and $D_i = \textit{at most } d_i$ on their ending dates thus forcing the activities to take place within intervals $[r_i, d_i]$. There may also be a global temporal feasibility window $[r, d]$ that constrains the whole set of activities. Suppose the processing times are known to belong to intervals P_i. The problem is now stated as follows: Find the starting times t_i of activities such that these activities take place within the feasiblity windows $[r_i, d_i]$ and the whole project starts after $t = r$ and ends not after $t = d$, *whatever the actual values of* $p_i \in P_i$ *turn out to be*. Clearly it comes down to solving a regular constrained scheduling problem, working with *maximal* values of $p_i \in P_i$ for the sake of robustness. These maximal values are pessimistic predictions of processing times.

The corresponding fuzzy optimal scheduling problem has been formulated in (Dubois et al., 1995). Consider a partially ordered set of activities. The performance of each activity i requires an ill-known processing time P_i like in Section 3.1. For simplicity, suppose there is only a global fuzzy release date and a global fuzzy due-date, respectively R and D like in Section 2. The formulation of the problem in terms of the pessimistic possibilistic preference functional extends the above non-fuzzy robust approach: find choices of starting times $t = (t_\alpha, \ldots, t_\omega)$ such that for any choice $p = (p_\alpha, \ldots, p_\omega)$ of processing times compatible with fuzzy intervals P_i, the fuzzy window constraints be satisfied to the best extent. The following problem can be stated:

$$\sup_{t_\alpha,\ldots,t_\omega} \inf_p \max(1 - \min_i \mu_{P_i}(p_i), \min(\mu_D(t_\omega), \mu_{R,}(t_\alpha)))$$

with $\max_{C \in C(\alpha,\omega)} \sum_{i \in C} p_i = t_\omega - t_\alpha$. Note that other starting times result from the determination of the launching time t_α and the ending time, as well as the processing times determined by solving the above problem.

The pessimistic preference functional is clearly seen in the "inf-max" expression. Letting $x = \max_{C \in C(\alpha,\omega)} \sum_{i \in C} p_i$ the objective function can be reformulated equivalently as

$$\sup_{t_\alpha,\ldots,t_\omega} \inf_x \max(1 - \sup_p \min_i \mu_{P_i}(p_i), \min(\mu_D(t_\omega), \mu_{R,}(t_\alpha)))$$

with $\max_{C \in \mathbf{C}(\alpha,\omega)} \sum_{i \in C} p_i = x = t_\omega - t_\alpha$. But, $\sup_p \min_i \mu_{P_i}(p_i)$ with constraint $\max_{C \in C(\alpha,\omega)} \sum_{i \in C} p_i = x$ yields the fuzzy duration of the project say a fuzzy number L. Now the problem reads

$$cons = \sup_{t_\alpha,t_\omega} \inf_x \max(1 - \mu_L(x), \min(\mu_D(t_\omega), \mu_{R,}(t_\alpha)))$$

with $x = t_\omega - t_\alpha$. A technical lemma can be established:

Lemma. *With continuous membership functions, and a bounded-support* L, *the following identity holds when* $x = t_\omega - t_\alpha$:

$$\sup_{t_\alpha,t_\omega} \inf_x \max(1 - \mu_L(x), \min(\mu_D(t_\omega), \mu_R(t_\alpha))) = \inf_x \max(1 - \mu_L(x), \mu_{D \ominus R}(x)).$$

Proof. Letting $t_\omega = x + t_\alpha$ it comes

$$
\begin{aligned}
cons &= \sup_{t_\alpha} \inf_x \max(1 - \mu_L(x), \min(\mu_D(x + t_\alpha), \mu_R(t_\alpha))) \\
&= \sup_{t_\alpha} \min(\inf_x \max(1 - \mu_L(x), \mu_D(x + t_\alpha)), \min(\inf_x \max(1 - \mu_L(x), \mu_R(t_\alpha)))) \\
&= \sup_{t_\alpha} \min(\inf_x \max(1 - \mu_L(x), \mu_D(x + t_\alpha)), \mu_R(t_\alpha)) \text{ since } \inf_x 1 - \mu_L(x) = 0.
\end{aligned}
$$

Moreover $\mu_D(x + t_\alpha)$ is a continuous decreasing function, and $1 - \mu_L(x)$ is a continuous function that decreases to 0 and then increases again to 1. Let l^* be a value in the core of L. The infimum of $\max(1 - \mu_L(x), \mu_D(x + t_\alpha))$ is attained for a value $x^* \geqslant l^*$ such that $1 - \mu_L(x^*) = \mu_D(x^* + t_\alpha)$. Hence this is also the supremum of $\min(1 - \mu_{]L,+\infty)}(x), \mu_D(x + t_\alpha))$, where the fuzzy threshold $]L, +\infty)$ has an increasing membership function equal to the fuzzy complement of the decreasing part of L. Hence we can compute

$$
\begin{aligned}
cons &= \sup_{t_\alpha} \min(\sup_x \min(1 - \mu_{]L,+\infty)}(x), \mu_D(x + t_\alpha)), \mu_R(t_\alpha)) \\
&= \sup_x \min(1 - \mu_{]L,+\infty})(x), \sup_{t_\alpha} \min(\mu_D(x + t_\alpha)), \mu_R(t_\alpha)) = \sup_x \min(1 - \mu_{]L,+\infty)}(x), \mu_{D_\Theta R}(x)) \\
&= \inf_x \max(1 - \mu_L(x), \mu_{D_\Theta R}(x)). \qquad \square
\end{aligned}
$$

The above lemma tells us that the degree of consistency of the problem is the degree of inclusion of the fuzzy length of the project in the fuzzy time window where the project must take place. This expression is a typical pessimistic preference functional. The proof is also instructive as it suggests that the practical solving of this problem comes down to one of simple scheduling with fuzzy constraints on ending and launching times and processing times as well. The latter fuzzy constraints on processing times are of the form $]P_i, +\infty)$ that is consider pessimistic (higher) values of processing times (see also Dubois et al., 1995). Hence the optimal solution can be obtained by the following procedure:

(1) Compute the fuzzy length L of the project by forward propagation of fuzzy durations $P'_i =]P_i, +\infty)$, which can be viewed as fuzzy pessimistic predictions.
(2) Compute the fuzzy span of the allowed horizon of the project. This is $D_\Theta R$.
(3) The degree of consistency of the problem is

$$cons = \sup_x \min(1 - \mu_{]L,+\infty)}(x), \mu_{D_\Theta R}(x)).$$

If $cons = 1$, the problem with durations equal to the maximal values of support(P_i) and strict constraints core(R) and core(D) is feasible. Hence the fuzzy problem is surely feasible.
(4) If $cons < 1$, defuzzify R by letting $t_\alpha = \mu_R^{-1}(cons)$.
(5) Compute pessimistic estimates (predictions) of durations as $p_i = \mu^{-1}_{]P_i,+\infty)}(cons)$ using the fuzzy complement of the decreasing side of P_i.

Clearly, we are back to the problem of flexible scheduling and we generalize the cautious version of constrained scheduling under interval-valued processing times outlined above.

Further iterations may be useful after step 5 so as to find the unique Pareto-optimal solution, as suggested by Dubois and Fortemps (1999). The case where local fuzzy due dates and release dates constrain the activities individually is a matter or further study (see Dubois et al., 1995).

The fact that the increasing parts of the fuzzy processing times play no role in pessimistic solution of the flexibly constrained scheduling problem under uncertainty should not be surprizing if we compare to its non-fuzzy version. Namely, consider a solution which prescribes starting dates t_i of activities i. A precedence constraint requiring activity i before activity k will be certainly verified only to a degree, due to ill-known processing times. This degree can be computed as:

$$N(p_i \leqslant t_k - t_i) = \inf\{1 - \mu_{P_i}(u) : p_i > t_k - t_i\} = \mu_{]P_i,+\infty)}(t_k - t_i). \tag{16}$$

Only an increasing membership function equal to the fuzzy complement of the decreasing part of p_i is used. This degree is all the smaller as interval $[t_i, t_k]$ is small, since the smaller this interval, the more unlikely the activity i can fit in.

Not so many papers consider flexibly constrained scheduling problems with fuzzy processing times of activities. The use of possibility theory in jobshop scheduling has been discussed Kerr and Walker (1989) and (Dubois, 1989), in which local evaluations are used to sequence operations on machines; Dubois et al. (1995) cast the above framework in the job-shop scheduling context and bridge the gap with constraint-directed propagation rules from Artificial Intelligence. Stanfield et al. (1996) compute the minimum ready time and optimal sequence of n jobs with fuzzy service times and due dates, by fixing acceptance levels. Litoiu and Tadei (2000) extend their approach to periodical real-time task scheduling so as to handle fuzzy processing times, by means of a fuzzy ranking method. The aim is to assign priorities to tasks so as to satisfy fuzzy due-dates.

4. Conclusion: The potential of possibility theory in scheduling problems

Possibility theory offers tools for a flexible extension of the constraint-directed approach to scheduling so as to introduce preference notions, as well as means of capturing uncertainty on data such as processing times, when this uncertainty stems from incomplete knowledge.

The flexible constraint approach has specificities which make it attractive in scheduling problems:

- It easily lends itself to the expression of non-compensatory local criteria. Flexible constraints may be more expressive than single global objective functions found in the literature. The minimization of the makespan can also be encoded as a particular fuzzy constraint.
- It presupposes that violating some constraints cannot be compensated by the satisfaction of other ones. The aim is not cost minimization, but a balanced handling local delays. This remark clearly tells the flexible constraint approach from additive multicriteria methodologies.
- It is in full agreement with the constraint-directed approach. Existing constraint propagation tools can be directly adapted, either by directly propagating preference profiles, or by solving a sequence of standard constraint satisfaction problems via level-cuts of the fuzzy sets obtained by fixing aspiration levels in a dichotomy technique.
- the complexity of the flexible constraint satisfaction method is not much higher than the complexity of standard constraint based methods, and can be applied to problems of similar size.

Actually, the optimal solutions to a flexible scheduling problem can be viewed as the feasible solutions of a regular constrained scheduling problem obtained by minimally relaxing the constraints defined by the cores of the fuzzy sets. This relaxation is determined by the level of consistency of the problem. The aim of a flexible scheduling problem is thus to achieve a trade-off between local specifications, so as to ensure the existence of a balanced solution. Flexible constraints enable two pitfalls of constraint-directed satisfaction to be obviated: the situation when the problem is overconstrained and the inconsistency is discovered after many computational steps, and the situation when the problem underconstrained and a solution is chosen at random when checking consistency. Some computational studies indicate that the introduction of local preference in constraint-directed scheduling often helps finding a better solution faster than standard constraint-based modelling (Fargier, 1994, 1997). More computational experiments need to be carried out to assess the practical benefits of flexibly constrained scheduling.

Possibility theory also provides a simple modelling of ill-known parameters, like activity durations. Possibilistic uncertainty analysis is an extension of sensitivity analysis to intervals of various confidence levels. It differs from a probabilistic analysis by not requiring extensive statistics nor independence assumptions. Possibility-based critical path analysis only encounters part of the computational difficulties met by the probabilistic approach. For instance the computation of the distribution of earliest starting times of activities and the ending time of a project is much simpler in the possibilistic approach than with a probabilistic model. However the accurate determination of distributions of latest starting times and floats and the criticality analysis is already a complex problem in the mere interval-valued case, which seems to be very little known, if ever studied at all. In particular, the standard view according to which critical activities form at least one critical path from the beginning task to the ending task must be given up, because there may exist isolated surely critical tasks, while a surely critical path may fail to exist (Chanas et al., 2002).

The joint handling of flexible constraints and uncertainty can be achieved in possibility theory using specific preference functionals that generalize the maximin and maximax criteria of decision-making under ignorance, focusing on pessimistic or optimistic plausible predictions of activity durations.

Appendix A. Basic notions of fuzzy set and possibility theory

Basic definitions of fuzzy set and possibility theory are given for a better understanding of the main text. More details can be found in Dubois and Prade (1980, 1988, 2000), for instance.

A fuzzy set A is a subset of a referential set U whose boundaries are gradual rather than abrupt. More formally: The *membership function* μ_A of a fuzzy set A assigns to each element $u \in U$ its degree of membership $\mu_A(u)$ usually taking values in $[0,1]$ (if the referential is numerical). The *core* of A is the set $c(A) = \{u, \mu_A(u) = 1\}$. It gathers the prototypes of A. The *height* $h(A)$ of a fuzzy set A is the maximal degree of membership of elements in U to A. If this height is less than 1, the core of A is empty. The cut of A at level α (or α-cut) is the standard (non-fuzzy, or crisp) set A_α of elements in U whose degree of membership to A is atleast α. These level-cuts form a family of nested sets which are the horizontal representation of a fuzzy set. The *support* of A is the set $s(A) = \{u, \mu_A(u) > 0\}$. It contains both the prototypes of A and its peripheral elements (elements of the boundary $0 < \mu_A(u) < 1$). Only prototypes of "*not A*" ($\mu_A(u) = 0$) are rejected.

The *complement* of the fuzzy set A in U is denoted A^c and its membership function is $\mu_A^c = 1 - \mu_A$. The *union* and *intersection* of fuzzy sets are obtained by respectively taking the maximum and the minimum of membership degrees of the each element of U in each of the fuzzy sets. These two operations commute with cuts.

A possibility distribution (Zadeh, 1978) is the membership function of a fuzzy set A, attached to single-valued variable x. It is denoted $\pi_x = \mu_A$ and represents the set of more or less plausible, mutually exclusive values of x. It is supposed that A is not empty, i.e. $\pi_x(u) = 1$ for at least one value u. A possibility distribution is similar to a probability density. However, $\pi_x(u) = 1$ only means that $x = u$ is a plausible situation, which cannot be excluded. A degree of possibility can be viewed as an upperbound of a degree of probability, at the mathematical level. A numerical possibility distribution encodes a family of probability functions.

Possibility theory encodes incomplete knowledge while probability theory accounts for random, accurately observed, phenomena or reflects a subjective betting behavior. In particular, the complete ignorance about x is expressed by $\pi_x(u) = 1$, for all $u \in U$. Numerical possibility theory is in agreement with probability theory, except for the Bayesian credo stating that any state of knowledge can be represented by a *single* probability distribution.

The *possibility* of an event "$x \in E$", denoted by $\Pi(x \in E)$ is the height of the intersection between the fuzzy set A such that $\pi_x = \mu_A$ and the set E:

$$\Pi(x \in E) = \sup_u \min(\pi_x(u), \mu_E(u)) = \sup_{u \in E} \pi_x(u).$$

This degree evaluates the extent to which "$x \in E$" is "possibly" true, or to what extent the proposition "$x \in E$" is consistent with the item of information "$x \in A$" modelled by $\pi_x = \mu_A$. Note that in the left part of the previous equation, E can be a fuzzy set.

The dual measure of necessity of "$x \in E$", denoted $N(x \in E)$ evaluates the extent to which the fuzzy set A is fully included in the core of E, in other words, to what extent the proposition "$x \in E$" is certainly true, i.e., implied by the item of information "$x \in A$":

$$N(x \in E) = \inf_u \max(1 - \pi_x(u), \mu_E(u)) = \inf_{u \notin E} 1 - \pi_x(u) = 1 - \Pi(x \in E^c)$$

where E^c is the complement of E. Indeed, $N(x \in E) = 1$ if and only if the support of A included in the core of $E : x \in E$ is sure if and only if *all* the more or less possible values of x are amidst the values fully satisfying the fuzzy constraint $x \in E$.

A fuzzy interval is a fuzzy set of the real line whose cuts are intervals. Cuts are closed intervals when the membership function is upper semi-continuous. The simplest representation of a fuzzy interval uses a trapezoidal membership function (Fig. 2) defined by linear interpolation from two nested intervals $[a,b] \subseteq [c,d]$ respectively forming the core and the support of the fuzzy set. A fuzzy interval with an increasing ($b = d = +\infty$) or a decreasing ($a = c = -\infty$) membership function is called a fuzzy threshold.

Given a real-valued variable x, a fuzzy interval A restricting the possible values of x several fuzzy thresholds $(-\infty, A]$, $(-\infty, A[$, $[A, +\infty)$, $]A, +\infty)$ can be defined:

$$\mu_{(-\infty,A]}(r) = \Pi(r \leqslant x) = \sup_{u \geqslant r} \mu_A(u),$$
$$\mu_{(-\infty,A[}(r) = N(r \leqslant x) = \inf_{u<r} 1 - \mu_A(u),$$
$$\mu_{[A,+\infty)}(r) = \Pi(r \geqslant x) = \sup_{u \leqslant r} \mu_A(u),$$
$$\mu_{]A,+\infty)}(r) = N(r \geqslant x) = \inf_{u>r} 1 - \mu_A(u),$$

which are membership functions of fuzzy sets of numbers respectively possibly less than x, certainly less than x, possibly greater than x and certainly greater than x (cf. Fig. 2).

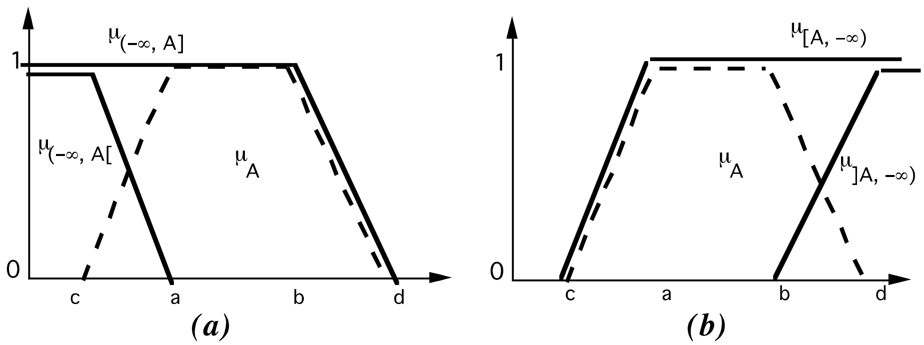

Fig. 2. (a) Numbers possibly/certainly less than A. (b) Numbers possibly/certainly greater than A.

Real-valued functions can be extended to interval arguments. Let A and B be two real-valued intervals restricting the possible values of two logically independent variables x and y. Given a two place function $f(x,y)$, $f(A,B)$ is defined to be $\{f(a,b), a \in A, b \in B\}$. If A and B are fuzzy intervals, the membership function of $f(A,B)$ is defined by the extension principle:

$$\mu_{f(A,B)}(z) = \sup_{x,y:z=f(x,y)} \min(\mu_A(x), \mu_B(y)).$$

It can be reconstructed via interval computation from cuts, since in general the α-cut of $f(A,B)$ is $f(A_\alpha, B_\alpha)$.

For instance, the sum $A \oplus B$ and difference $A\Theta B$ of two fuzzy intervals are such that:

$$\mu_{A\oplus B}(z) = \sup_x \min(\mu_A(x), \mu_B(z-x)),$$
$$\mu_{A\Theta B}(z) = \sup_x \min(\mu_A(x), \mu_B(z+x)).$$

Similarly, the possibility distributions attached to $\min(x,y)$ and $\max(x,y)$ changing f into the minimum and the maximum can be derived via the extension principle. As shown on Fig. 3 the fuzzy interval $\widetilde{\max}(A,B)$ may be different from both A and B, and similarly for $\widetilde{\min}(A,B)$. The extended minimum and maximum satisfy most usual other properties for instance,

$$\widetilde{\min}(A,A) = \widetilde{\max}(A,A) = A \text{ (idempotence)},$$
$$\widetilde{\max}(A,B) \oplus C = \widetilde{\max}(A \oplus C, B \oplus C),$$

or yet the mutual distributivity of $\widetilde{\min}$ and $\widetilde{\max}$. However, computing the possibility distribution induced by more general functions can be difficult when these functions are not monotonic, or when the variables restricted by fuzzy intervals are linked. Applying extended basic operations like sums and products to compute $f(A,B)$ is not always possible even if function f can be expressed as such (like for the latest starting times and floats of activities in Section 3).

Comparing fuzzy intervals is different from computing the extended minimum and maximum. It is a controversial matter, as witnessed by an abundant and heterogeneous literature. (See Bortolan and Degani, 1985; Chen and Hwang, 1992; Dubois et al., 2000 for surveys.) Only the most widely acknowledged techniques are recalled here.

Two situations must be distinguished: the case when the aim is to get a complete ranking of fuzzy intervals, and the case when the aim is just to describe their relative position. For instance, the problem of finding the best sequencing of activities by ranking fuzzy makespans pertains to the first situation. The second situation makes sense if the aim is only to know degrees of possible or sure dominance between fuzzy schedules so as to estimate their relative worth.

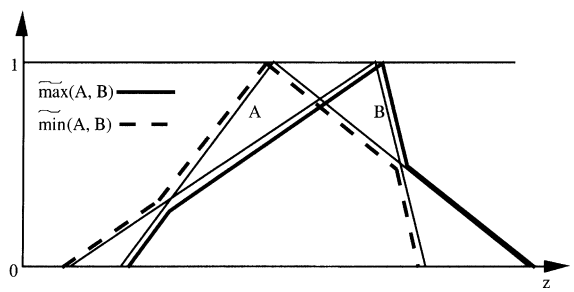

Fig. 3. Maximum and minimum of two fuzzy intervals A and B.

Ranking fuzzy intervals is usually done via so-called defuzzification. The name ''defuzzification'' is misleading. Strictly speaking it means ''making a fuzzy set not fuzzy''; it should yield a set. However, people use the word to mean ''find a scalar substitute to a fuzzy quantity''. There are many defuzzification procedures, which have been proposed without other convincing motivation than practical convenience. Ranking fuzzy intervals then comes down to ranking their scalar substitutes. However, one of them, proposed by Yager (1981) turned out to be the most natural, although not so often used: the neutral scalar substitute $s(A)$ of a fuzzy interval A is defined by

$$s(A) = \frac{1}{2}\int_0^1 (a_\alpha^- + a_\alpha^+)\mathrm{d}\alpha$$

where $[a_\alpha^-, a_\alpha^+]$ is the α-cut of A. This definition can be retrieved using the area compensation method (Fortemps and Roubens, 1996). Considering the set of all probability functions dominated by the possibility function induced by μ_A, $s(A)$ is also the expectation of the probability distribution which lies at the center of gravity of that set. The major property of $s(A)$ is its linearity: $s(A \oplus B) = s(A) + s(B)$ and $s(\rho \cdot A) = \rho s(A)$, for any real ρ.

More generally, alternative scalar substitutes of A preserving linearity can be obtained by using a convex combination $s_\lambda(A)$ obtained by the integral of $\lambda a_\alpha^- + (1-\lambda)a_\alpha^+$, where λ is a degree of pessimism, prescribed by the user, like in Hurwicz criterion in decision-making. The interval $[s_1(A), s_0(A)]$ is the mean interval of A (Dubois and Prade, 1987). It is the range of the expectations of probability functions dominated by the possibility function induced by μ_A and its middle point is $s(A)$.

If the problem is to describe the relative location of two fuzzy intervals A and B, one may compute the possibility that there exists a value of x smaller than A and greater than B (Baas and Kwakernaak, 1977):

$$\Pi(A \geqslant B) = \sup_x \min(\mu_{(-\infty,A]}(x), \mu_{[B,+\infty)}(x)).$$

This purely metrical index, and its dual one $N(A > B) = 1 - \Pi(B \geqslant A)$ are the direct fuzzy extension of interval orderings based on the following idea: $[a,b] \geqslant [c,d]$ if and only if $a \geqslant d$. Then, $\Pi([c,d] \geqslant [a,b]) = 0$ and $\Pi([a,b] \geqslant [c,d]) = 1$. If $[a,b] \cap [c,d] \neq \emptyset$, then $\Pi([c,d] \geqslant [a,b]) = \Pi([a,b] \geqslant [c,d]) = 1$, a form of indifference. The properties of this index extend to the valued case the properties of interval orderings (Fodor and Roubens, 1994; Pirlot and Vincke, 1997).

Lastly, possibility theory offers a framework similar to, but distinct from, utility theory for decision-making under uncertainty. Restricting ourselves to the classical decision-theoretic setting, where a decision is modelled as a function δ from a state space S to a set of consequences X, the knowledge of the system state is supposed to be defined by a possibility distribution π on S, the decision-maker preferences among consequences are described by another function $\mu : X \to [0,1]$ ($\mu(x)$ is a kind of utility degree of consequence x). Then two preference functionals construed as possibility and necessity evaluations can be used for rating a decision δ:

- a pessimistic function: $u_*(\delta) = \inf_{s\in S} \max(1 - \pi(s), \mu(\delta(s)))$;
- an optimistic function: $u^*(\delta) = \sup_{s\in S} \min(\pi(s), \mu(\delta(s)))$.

Noticing that a state s such that $\pi(s) = 1$ is viewed as a plausible state, $u_*(d)$ measures the extent to which decision δ has good consequences in all plausible states. In particular if the knowledge is absent then $\pi(s) = 1, \forall s$, and $u_*(\delta) = \inf_{s\in S} \mu(\delta(s))$; this is Wald pessimistic criterion which is obtained as a special case. On the contrary, $u^*(\delta)$ is high if there exists a plausible state where decision δ produces a good consequence. These criteria differ from expected utility based on a probability distribution Prob on states, of the form $u(\delta) = \Sigma_s \mathrm{Prob}(s) \cdot \mu(d(s))$, for two reasons:

- they presuppose less information about the state;
- they make sense for one-shot decisions, or successive decisions whose individual utilities do not cumulate (unlike costs, for instance).

The rationality of these criteria and of possibility theory as an observable uncertainty theory has been laid bare via an act-based axiomatisation in the Savage style (Dubois et al., 2001), in a finite setting.

References

Baas, S.M., Kwakernaak, H., 1977. Rating and ranking of multiple aspect alternatives using fuzzy sets. Automatica 13, 47–58.

Bellman, R.E., Zadeh, L.A., 1970. Decision-making in a fuzzy environment. Management Science 17 (4), B141–B164.

Bensana, E., Bel, G., Dubois, D., 1988. OPAL, a multi-knowledge-based system for industrial job-shop scheduling. International Journal of Production Research 26 (5), 795–816.

Birge, J.R., Dempster, M.A.H., 1996. Stochastic programming approaches to stochastic scheduling. Journal of Global Optimization 9, 383–409.

Blazewicz, J., Cellary, W., Slowinski, R., Weglarz, J. (Eds.) 1986. Scheduling under resource constraint—Deterministic Models. Annals of Operations Research, vol. 7, J.C. Baltzer AG.

Bortolan, G., Degani, R., 1985. A review of some methods for comparing fuzzy subsets. Fuzzy Sets and Systems 15, 1–19.

Buckley, J.J., 1989. Fuzzy PERT. In: Evans, G., Karwowski, W., Wilhelm, M. (Eds.), Applications of the Fuzzy Set Methodologies in Industrial Engineering. Elsevier, pp. 103–114.

Chanas, S., Kamburowski, J., 1981. The use of fuzzy variables in PERT. Fuzzy Sets and Systems 5, 11–19.

Chanas, S., Kuchta, D., 1998. Discrete fuzzy optimization. In: Slowinski, R. (Ed.), Fuzzy Sets in Decision Analysis, Operations Research and Statistics. The Handbooks of Fuzzy Sets Series. Kluwer Academic Publishers, Boston, pp. 249–277.

Chanas, S., Zielinski, P., 2002. The computational complexity of interval critical path method. European Journal of Operational Research 136 (3), 541–550.

Chanas, S., Zielinski P., 2001. Critical path analysis in the network with fuzzy activity times, to appear in Fuzzy Sets and Systems, 122 (2), 195–204.

Chanas, S., Dubois, D., Zielinski, P., 2002. Necessary criticality in the network with imprecise activity times. IEEE Transactions on Systems, Man and Cybernetics 32, 393–407.

Chen, S.-J., Hwang, C.-L., 1992. In: Fuzzy Multiple Attribute Decision Making, Lecture Notes in Economics and Mathematical Systems, vol. 375. Springer Verlag, Berlin.

Dubois, D., 1983. Modèles mathématiques de l'imprécis et de l'incertain en vue d'applications aux techniques d'aide á la décision, Thèse d'état de l'Université Scientifique et Médicale de Grenoble.

Dubois, D., 1989. Fuzzy knowledge in an artificial intelligence system for job-shop scheduling. In: Evans, G., Karwowski, W., Wilhelm, M. (Eds.), Applications of the Fuzzy Set Methodologies in Industrial Engineering. Elsevier, pp. 73–89.

Dubois, D., Fortemps, P., 1999. Computing improved optimal solutions to fuzzy constraint satisfaction problems. European Journal of Operational Research 118, 95–126.

Dubois, D., Koning, J.L., 1994. A decision engine based on rational aggregation of heuristic knowledge. Decision Support Systems 11, 337–361.

Dubois, D., Prade, H., 1978. Algorithmes de plus courts chemins pour traiter des donnees floues. RAIRO-Recherche Opérationnelle/Operations Research 12, 212–227.

Dubois, D., Prade, H., 1980. Fuzzy Sets and Systems—Theory and Applications. Academic Press, New York.

Dubois, D., Prade, H., 1987. The mean value of a fuzzy number. Fuzzy Sets and Systems 24, 279–300.

Dubois, D., Prade, H., 1988. Possibility Theory. Plenum Press, New York.

Dubois, D., Prade, H. (Eds.), 2000. Fundamentals of Fuzzy Sets. In: The Handbooks of Fuzzy Sets Series. Kluwer Academic Publishers, Boston.

Dubois, D., Fargier, H., Prade, H., 1995. Fuzzy constraints in job-shop scheduling. Journal of Intelligent Manufacturing 6, 215–234.

Dubois, D., Fargier, H., Prade, H., 1996. Possibility theory in constraint satisfaction problems: Handling priority, preference and uncertainty. Applied Intelligence 6, 287–309.

Dubois, D., Kerre, E., Mesiar, R., Prade, H., 2000. Fuzzy interval analysis. In: Dubois, D., Prade, H. (Eds.), Fundamentals of Fuzzy sets. The Handbooks of Fuzzy Sets Series. Kluwer Academic Publishers, Boston, pp. 483–581.

Dubois, D., Fargier, H., Galvagnon, V., 2001. On latest starting times and floats in activity networks with ill-known durations. European Journal of Operational Research 147 (2), this issue.

Dubois, D., Prade, H., Sabbadin, R., 2003. Decision-theoretic foundations of qualitative possibility theory. European Journal of Operational Research 128, 459–478.

Erschler, J., Esquirol, P., 1986. Decison-aid in job shop scheduling: A knowledge based approach. In: Proceedings of the 1986 IEEE Conference on Robotics and Automation, San Franscisco, pp. 1751–1756.
Erschler, J., Roubellat, F., Vernhes, J.P., 1976. Finding some essential characteristics of the feasible solutions for a scheduling problem. Operations Research 24, 774–783.
Esquirol, P., Lopez, P., 1999. L'ordonnancement. Economica, Paris.
Fargier, H., 1994. Problèmes de satisfaction de contraintes flexibles: Application à l'ordonnancement de production. Thèse de Doctorat, Université Paul Sabatier, Toulouse.
Fargier, H., 1997. Fuzzy scheduling: Principles and experiments. In: Dubois, D., Prade, H., Yager, R.R. (Eds.), Fuzzy Information Engineering: A Guided Tour of Applications. Wiley, New York, pp. 655–669.
Fargier, H., Galvagnon, V., 1999. Le Problme du PERT flou dans les graphes série-paralléle. In: Actes des Rencontres Francophones sur la Logique Floue et ses Applications (LFA'99), Valenciennes. Cepadues, Toulouse, pp. 277–284.
Fargier, H., Galvagnon, V., Dubois, D., 2000. Fuzzy PERT in series–parallel graphs. In: Ninth IEEE International Conference on Fuzzy Systems, San Antonio, Texas, pp. 717–722.
Fodor, J., Roubens, M., 1994. Fuzzy Preference Modelling and Multicriteria Decision Support. Kluwer Academic Publishers, Dordrecht, section 7.2.2.
Fortemps, P., 1997. Job-shop scheduling with imprecise durations: A fuzzy approach. IEEE Transactions on Fuzzy Systems 5, 557–569.
Fortemps, P., Hapke, M., 1997. On the disjunctive graph for project scheduling. Foundations of Computing and Decision Sciences 22, 195–209.
Fortemps, P., Roubens, M., 1996. Ranking and defuzzification methods based on area compensation. Fuzzy Sets and Systems 82, 319–330.
Fox, M.S., 1987. Constraint-Directed Search: A Case Study of Job-Shop Scheduling. Pitman, London.
Gazdik, I., 1983. Fuzzy network planning FNET. IEEE Transactions on Reliability 32, 303–313.
Grabot, B., Geneste, L., 1994. Dispatching rules in scheduling: A fuzzy approach. International Journal of Production Research 32, 903–915.
Guiffrida, A.L., Nagi, R., 1998. Fuzzy set theory applications in production management research: A literature survey. Journal of Intelligent Manufacturing 9, 39–56.
Hapke, M., Jaszkiewicz, A., Slowinski, R., 1994. Fuzzy project scheduling system for software development. Fuzzy Sets and Systems 21, 101–117.
Hintz, G.W., Zimmermann, H.-J., 1989. A method to control flexible manufacturing systems. European Journal of Operational Research 41, 321–334.
Ishibuchi, H., Yamamoto, N., Misaki, S., Tanaka, H., 1994. Local search algorithms for flowshop scheduling with fuzzy due-dates. International Journal of Production Economics 33, 53–66.
Ishii, H., Tada, M., Masuda, T., 1992. Two scheduling problems with fuzzy due-dates. Fuzzy Sets and Systems 46, 339–347.
Kaufmann, A., Gupta, M.M., 1988. Fuzzy Mathematical Models in Engineering and Management Science. Amsterdam, North Holland.
Kerr, R.M., Walker, R.N., 1989. A job shop scheduling system based on fuzzy arithmetics. In: Proceedings of the third International Conference on Expert Systems and the Leading Edge in Production and Operation Management, Hilton Head, SC pp. 433–450.
Lin, K.C., Chern, M.S., 1993. The fuzzy shortest path problem and its most vital arcs. Fuzzy Sets and Systems 58, 343–353.
Litoiu, M., Tadei, R., 1997. Real-time task scheduling allowing fuzzy due-dates. European Journal of Operational Research 100, 475–481.
Litoiu, M., Tadei, R., 2000. Real-time scheduling with fuzzy deadlines and processing times. Fuzzy Sets and Systems 117, 35–46.
Lootsma, F., 1989. Stochastic and fuzzy PERT. European Journal of Operational Research 43, 174–183.
Lootsma, F., 1997. Fuzzy Logic for Planning and Decision-Making. Kluwer Academic Publishers, Dordrecht.
MacCahon, C.S., 1993. Using PERT as an approximation of fuzzy project network analysis. IEEE trans. Engineering Management 40, 146–153.
MacCahon, C.S., Lee, E.S., 1988. Project network analysis with fuzzy activity times. Computers and Mathematics with Applications 15, 829–838.
MacCahon, C.S., Lee, E.S., 1992. Fuzzy job sequencing for a flow shop. European Journal of Operational Research 62, 294–301.
MacCarthy, B., Liu, J., 1993. Addressing the gap in scheduling research: A review optimization and heuristic methods in production scheduling. International Journal of Production Research 31, 59–79.
Mares, M., 1989. Network analysis of fuzzy technologies. In: Evans, G., Karwowski, W., Wilhelm, M. (Eds.), Applications of the Fuzzy Set Methodologies in Industrial Engineering. Elsevier, pp. 115–125.
Nasution, S.H., 1993. Fuzzy critical path method. IEEE Transactions on Man, Machine and Cybernetics 24 (1), 48–57.
Özelkan, E.C., Duckstein, L., 1999. Optimal fuzzy counterparts of priority rules. European Journal of Operational Research 113, 593–609.
Pirlot, M., Vincke, P., 1997. Semiorders: Properties, representations, applications. Kluwer Academic Publishers, Dordrecht.

Prade, H., 1979. Using fuzzy set theory in a scheduling problem: A case study. Fuzzy Sets and Systems 2, 153–165.

Rommelfanger, H.J., 1990. FULPAL: An interactive method for solving (multiobjective) fuzzy linear programming problems. In: Slowinski, R., Teghem, J. (Eds.), Stochastic versus Fuzzy Approaches to Multiobjective Mathematical Programming under Uncertainty. Kluwer, pp. 321–330.

Rommelfanger, H.J., 1994. Network analysis and information flow in fuzzy environment. Fuzzy Sets and Systems 67, 119–128.

Sadeh, N., Fox, M., 1996. Variable and value ordering heuristics for the jobshop scheduling constraint-satisfaction problem. Artificial Intelligence 86, 1–41.

Slany, W., 1996. Scheduling as a fuzzy multiple criteria decision problem. Fuzzy Sets and Systems 78, 197–222.

Slowinski, R., Hapke, M. (Eds.), 2000. Scheduling under Fuzziness. Physica-Verlag, Heidelberg, Germany.

Stanfield, P., King, R., Jones, J., 1996. Scheduling arrivals to a production system in a fuzzy environment. European Journal of Operational Research 93, 75–87.

Turksen, I.B., Fazel Zarandi, M.H., 1999. Production planning and scheduling: Fuzzy and crisp/approaches. In: Zimmermann, H.-J. (Ed.), Practical Applications of Fuzzy Technologies. The Handbook of Fuzzy sets Series. Kluwer Academic Publishers, Dordrecht, pp. 479–530.

vanLaarhoven, P.J.M., Aarts, E.H.L., Lenstra, J.K., 1992. Job-shop scheduling by simulated annealing. Operations Research 40, 112–125.

Wang, H.F., Fu, C.C., 1996. Fuzzy project scheduling models under inflation conditions. International Journal of Uncertainty, Fuzziness and Knowledge-based Systems 4, 514–597.

Werners, B., Weber, R., 1999. Decision and planning in research and development. In: Zimmermann, H.-J. (Ed.), Practical Applications of Fuzzy Technologies. The Handbook Fuzzy sets Series. Kluwer Academic Publishers, Dordrecht, pp. 445–478.

Yager, R.R., 1981. A procedure for ordering fuzzy subsets of the unit interval. Information Sciences 24, 143–161.

Zadeh, L.A., 1975. Calculus of fuzzy restrictions. In: Zadeh, L.A., Fu, K.S., Tanaka, K., Shimura, M. (Eds.), Fuzzy Sets and their Applications Cognitive and Decision Processes. Academic Press, New York, pp. 1–39.

Zadeh, L.A., 1978. Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets and Systems, 13–28.

Zimmermann, H.-J., 1985. Applications of fuzzy set theory to mathematical programming. Information Sciences 36, 28–58.